

REPRESENTATIONS OF MCLAIN GROUPS

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ABSTRACT. Basic modules of McLain groups $M = M(\Lambda, \leq, R)$ are defined and investigated. These are (possibly infinite dimensional) analogues of André's supercharacters of $U_n(q)$. The ring R need not be finite or commutative and the field underlying our representations is essentially arbitrary: we deal with all characteristics, prime or zero, on an equal basis. The set Λ , totally ordered by \leq , is allowed to be infinite. We show that distinct basic modules are disjoint, determine the dimension of the endomorphism algebra of a basic module, find when a basic module is irreducible, and exhibit a full decomposition of a basic module as direct sum of irreducible submodules, including their multiplicities. Several examples of this decomposition are presented, and a criterion for a basic module to be multiplicity-free is given. In general, not every irreducible module of a McLain group is a constituent of a basic module.

1. INTRODUCTION

In 1954 McLain [M] constructed a family of groups that has been a rich source of examples in group theory ever since (see [M2], [R], [HH], [Ro], [W], [DG], [CS], [Sz2], for instance). A general McLain group $M = M(\Lambda, \leq, R)$ depends on a set Λ , partially ordered by \leq , and an arbitrary ring R with $1 \neq 0$. Even though a partial order will do for some of our purposes, for best results a total order will be required. In the special case when $|\Lambda| = n$ is finite and \leq is a total order, $M = U_n(R)$ is the subgroup of $GL_n(R)$ of all upper triangular matrices with 1's on the main diagonal.

The main goal of this paper is define and study basic modules of M , which are a generalization of the supercharacters of $U_n(q)$, where $R = F_q$ is a finite field of characteristic p . We stress the fact that Λ as well as R are allowed to be infinite, and M -modules are allowed to be infinite dimensional over an arbitrary field F (which need not have characteristic 0). Moreover, the commutativity or not of R plays no role whatsoever, so we will allow R to be non-commutative. It is perhaps surprising how of much of the theory of supercharacters goes through in this context. A detailed description appears below, after an overview of prior work on the subject.

The representation theory of $U_n(q)$ draws considerable attention due to its attractive nature and open problems. The literature on the subject, as well as on the related algebra groups and Sylow p -subgroups of classical groups, is too vast to review in full detail and we will restrict ourselves to a limited overview.

One line of investigation was concerned with the degrees of the complex irreducible characters of $U_n(q)$. In 1974 Lehrer [L] considered the so called elementary characters of $U_n(q)$ as well as certain products of them, obtaining ([L, Corollary

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5.2']) irreducible characters of $U_n(q)$ of degree q^c for every integer c such that $0 \leq c \leq \mu(n) = (n-2) + (n-4) + \cdots$. Two decades later, Isaacs [I] confirmed that every irreducible character of, not only $U_n(q)$, but also every F_q -algebra group, has q -power degree. Isaacs' paper left open the question as to whether an earlier assertion by Gutkin was true: is every irreducible character of an algebra group induced from a linear character of an algebra subgroup? This was partially confirmed by André [A4] and fully by Halasi [H]. An extension of Halasi's result has recently been obtained by Boyarchenko [B]. Isaacs' result on character degrees does not extend directly to Sylow p -subgroups of other classical groups, a fact recognized by Isaacs himself and further confirmed by Gow, Marjoram and Previtali [GMP]. However, for p odd, Isaacs' result was successfully extended to Sylow p -subgroups of symplectic, orthogonal and unitary groups by Szegedy [Sze].

It is worth noting that all of Lehrer's elementary characters can be constructed by lifting those associated to the position $(1, n)$ and that, when q is odd, a character attached to this position is nothing but a Weil character of $H \rtimes U_{n-2}(q)$, where H is the Heisenberg group associated to a symplectic space of dimension $2(n-2)$ over F_q and $U_{n-2}(q)$ is viewed as a subgroup of a Sylow p -subgroup of the corresponding symplectic group. Moreover, this identification remains valid when F_q is replaced by more general finite, commutative, local rings of odd characteristic (see [CMS] for the Weil representation in this context).

Another source of research is related to a conjecture by Higman [Hi], to the effect that the number of irreducible characters of $U_n(q)$ is an integer polynomial in q . A sharpening of Higman's conjecture, attributed by Lehrer [L] to J.G. Thompson, states that, for $0 \leq c \leq \mu(n)$, the number of irreducible characters of $U_n(q)$ of degree q^c is an integer polynomial in q . A further sharpening of the latter was conjectured by Isaacs [I2], with evidence that the number of irreducible characters of $U_n(q)$ of degree q^c is a polynomial in $q-1$ with non-negative integer coefficients. Work on these conjectures has been intense. We refer the reader to [G], [Le], [Lo], [Me], [Me2], [Mm], [HP], [T], [VA] and [VA2] for work in this direction.

Rather than the degree of the irreducible characters of $U_n(q)$ or the total number of them, our attention is more related to the study of supercharacters of $U_n(q)$ and their irreducible constituents. As mentioned above, Lehrer [L] proved that certain products of elementary characters of $U_n(q)$ remain irreducible. This prompted André [A] to consider more general products of elementary characters, then called basic characters and now called supercharacters after the work of Diaconis and Isaacs [DI], who axiomatized a theory of supercharacters and applied it to algebra groups. The main result of [A] is that every irreducible character of $U_n(q)$ is a constituent of one and only one supercharacter. In particular, distinct supercharacters are orthogonal. A supercharacter need not be irreducible and André [A] gives a formula for the inner product of a supercharacter with itself. All of this was done under the assumption $p \geq n$, a restriction that was later removed in [A2]. The problem of "finding" the irreducible constituents of a supercharacter, as well as their multiplicities, was addressed by André in [A3, Theorem 2], who also gave [A3, Theorem 4] necessary and sufficient conditions for a supercharacter to be a multiple of an irreducible character. André and Nicolás [AN] produced a fairly wide generalization of the theory by considering not just $U_n(q)$ or even F_q -algebra groups but the adjoint group $G(A) = \{1 + a \mid a \in A\}$ of a finite nilpotent ring A . They were able to extend to this context the aforementioned result of Halasi. Moreover,

they defined and studied supercharacters in this context, obtaining a decomposition (cf. [AN, Theorem 4.2]) much like the one given in [A3, Theorem 2], as well necessary and sufficient conditions (cf. [AN, Theorem 6.1]) for a supercharacter to be a multiple of an irreducible character. At the end of [AN] they specialize to $U_n(q)$, reprove the formula for the inner product of a supercharacter with itself, and derive an irreducibility criterion for supercharacters (cf. [AN, Theorem 7.1]).

Before we state our main results, we must describe our overall assumption on R and F . Our only assumption is the existence of a right primitive linear character $R^+ \rightarrow F^*$, that is, a group homomorphism $R^+ \rightarrow F^*$ having no non-zero right ideals in its kernel. When R is finite this condition has been studied in detail (see [La], [CG], [Wo] and [Ho]): it is left and right symmetric, and equivalent to R being a Frobenius ring. This symmetry was left as an open question in [CG], although it had already been established in [La]. It is reproved in [Wo] and [Ho]. As a byproduct of our study of elementary modules, §4 furnishes an independent proof of this symmetry by means of fully ramified characters. The condition that R have a primitive linear character when R is a finite, local and commutative ring has already appeared in representation theory, e.g., in the context of the Weil representation of symplectic groups over rings (see [CMS2] and [Sz3]).

Let K be a non-archimidean local field with ring of integers \mathcal{O} , maximal ideal \mathfrak{p} and residue field $F_q = \mathcal{O}/\mathfrak{p}$ of prime characteristic p . Then $R = \mathcal{O}/\mathfrak{p}^m$ is a finite, principal, local, commutative ring of size q^m affording a primitive linear character when F has a root of unity of order p^m if $\text{char}(K) = 0$ and p if $\text{char}(K) = p$. This can arguably be considered as the most important example. A non-commutative analogue can be obtained from Hilbert's twist $\mathcal{O} = D[[x; \sigma]]$, the ring of skew power series twisted by an automorphism σ of a division ring D . This is a local ring with Jacobson radical $\mathfrak{p} = (x)$. Then $R = \mathcal{O}/\mathfrak{p}^m$ affords a primitive linear character if and only if so does D . In this regard, if D has prime characteristic p it suffices that F have a root of unity of order p , while if $\text{char}(D) = 0$ it is enough that, for some prime p , F have roots of unity of order p^ℓ for all $\ell \geq 1$. While in the above examples R is always a principal ring, this need not be the case. See [CG] for further examples. The actual choice of R has no effect whatsoever on our arguments.

We begin our paper with a reminder, in §2, of the construction and basic properties of McLain groups. Basic tools to deal with finite and infinite dimensional modules on an equal basis are found in §3, which essentially reproduces some of the results from [Sz] on the Clifford theory of possibly infinite dimensional modules. In §4 we define and study elementary modules of certain subgroups of M , which are themselves McLain groups, generalizing the elementary modules constructed by Lehrer. Our elementary modules give rise to the simplest example of two curious phenomena: an irreducible module having no irreducible submodules (and hence failing to be completely reducible) when restricted to a normal subgroup (further examples can be found [Sz]), and a family of (infinitely many) commuting diagonalizable operators (acting on an infinite dimensional vector space) having no common eigenvector.

Our paper properly begins after the above preliminary sections. From §5 onwards we must assume that \leq is a total order. In particular, this allows us to extend to all of M the action on elementary modules. Let

$$\Phi = \{(\alpha, \beta) \in \Lambda \times \Lambda \mid \alpha < \beta\}.$$

For $\alpha < \beta$ in Λ we will abuse notation and denote by (α, β) not just the pair in Φ , but also the open interval $\{\gamma \in \Lambda \mid \alpha < \gamma < \beta\}$ in Λ . To every triple (α, β, λ) , where $(\alpha, \beta) \in \Phi$ and $\lambda : R^+ \rightarrow F^*$ is a right primitive linear character, there corresponds an elementary M -module $V(\alpha, \beta, \lambda)$. This is finite dimensional over F if and only if the open interval (α, β) is empty, in which case $\dim(V(\alpha, \beta, \lambda)) = 1$, or R and (α, β) are both finite, in which case $\dim(V(\alpha, \beta, \lambda)) = |R|^{|\alpha, \beta|}$. Let

$$D = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$$

be a basic subset of Φ , in the sense that all α_i (resp. all β_i) are distinct. Let f be a choice function, from D into the set of all right primitive linear characters $R^+ \rightarrow F^*$, say $f(\alpha_i, \beta_i) = \lambda_i$, $1 \leq i \leq m$. Then the basic M -module $V(D, f)$ is

$$V(D, f) = V(\alpha_1, \beta_1, \lambda_1) \otimes \cdots \otimes V(\alpha_m, \beta_m, \lambda_m).$$

This is in complete agreement with André's basic characters, as defined in [A].

Given a group G and G -modules X and Y , we define

$$(X, Y)_G = \dim_F(\text{Hom}_{FG}(X, Y)).$$

Our first main result, Theorem 5.2, states that basic modules are disjoint, that is,

$$(1.1) \quad (V(D, f), V(D', f'))_M = 0 \text{ if } (D, f) \neq (D', f').$$

In words, distinct basic modules can only be connected via a zero homomorphism. We stress once more that this result as well as all results stated below are, until further notice, valid in utter generality: R and Λ are allowed to be infinite and a basic M -module may be infinite dimensional. As is often the case, a proof given in a more general context is conceptually simpler (as ideas are stripped to the bone) and we believe this is the case with Theorem 5.2 as well as others in this paper.

Let

$$\Omega = \{(\alpha, \gamma) \in \Phi \mid \exists (\alpha, \beta), (\gamma, \delta) \in D \text{ such that } \alpha < \gamma < \beta < \delta\}.$$

Then Theorem 5.9 gives the following irreducibility criterion for basic modules: $V(D, f)$ is irreducible if and only if $\Omega = \emptyset$. This is in complete agreement with the corresponding result for $U_n(q)$ (cf. [AN, Theorem 7.1]). Our proof of this irreducibility criterion is a direct application of a well-known theorem of Gallagher [Ga, Theorem 2]. No calculations of any kind are required. However, for full generality, we must resort to the extension of Gallagher's theorem found in [Sz, Theorem 3.11] (this extension is not required for those interested only in $U_n(q)$, or even $U_n(R)$ with R finite).

As in the case of $U_n(q)$, associated to each closed subset Γ of Φ there is a corresponding pattern subgroup $M(\Gamma)$ of M . Now D gives rise to the closed subset

$$\Gamma = \bigcap_{1 \leq i \leq m} (\Phi \setminus [\alpha_i, \rightarrow, \beta_i)) = \Phi \setminus \left(\bigcup_{1 \leq i \leq m} [\alpha_i, \rightarrow, \beta_i) \right),$$

where for $(\alpha, \beta) \in \Phi$ we define

$$[\alpha, \rightarrow, \beta) = \{(\alpha, \gamma) \in \Phi \mid \gamma < \beta\}.$$

If we let $H = M(\Gamma)$ then, much as in the case of $U_n(q)$, there is a linear character $\lambda : H \rightarrow F^*$, corresponding to $(\lambda_1, \dots, \lambda_m)$, and a 1-dimensional H -module W upon which H acts via λ , such that (cf. Theorem 5.4):

$$V(D, f) \cong \text{ind}_H^M W.$$

Again, this is in line with the corresponding result for $U_n(q)$, as found in [A]. Let

$$\Gamma_1 = \Gamma \cup \Omega \text{ (disjoint union),}$$

which is a closed subset of Φ , and let $I = M(\Gamma_1)$. Complementing (1.1) we have the following result (cf. Theorem 8.1):

$$(1.2) \quad (V(D, f), V(D, f))_M = [I : H].$$

This is the perfect analogue for André formula (cf. [A2, Theorem 2]) for the inner product of a basic character with itself (see also [Le2, Corollary 2.10]). Thus, $V(D, f)$ is irreducible if and only if $I = H$ (cf. Theorem 8.3).

The most delicate part of our paper deals with the actual decomposition of a basic character. This requires considerably more effort than above. The most significant obstacle we face is that a G -module X may satisfy $(X, X)_G = 1$ without being irreducible. Obviously, irreducibility in this setting amounts to complete reducibility. When R and Λ are finite this is a non-issue, since the existence of a primitive linear character implies $\text{char}(F) \nmid |R|$ (cf. Lemma 4.2), and complete reducibility follows from Maschke's theorem. By keeping R finite but allowing Λ to be arbitrary, we were able to overcome this obstacle by using the concept of ascendant subgroup. This is the same tool successfully used by Meierfrankenfeld [Me] and Wehrfritz [We]: if G is an irreducible subgroup of $\text{FGL}(V)$ (the full finitary general linear group) and H is an ascendant subgroup of G then H is completely reducible (in this regard, recall the aforementioned failure of a normal subgroup of an irreducible group to be completely reducible, encountered in §4). We actually require the opposite direction, from H to G , and have no need to resort to $\text{FGL}(V)$. Our results in this context are fairly general criteria for complete reducibility, found in Theorem 6.5 and its consequence, Theorem 6.7. The latter is a direct generalization of the well-known irreducibility criterion of Mackey [Ma], originally proved in the context of finite groups and finite dimensional modules over an algebraically closed field.

In order to be able to use the above general tools to decompose basic modules of McLain groups, it is necessary to verify the hypothesis these tools require. Much of this verification is carried out in §7 as well as in Theorem 8.7.

The above verification allows us to reach our main result, namely Theorem 8.8, which gives a full decomposition of a basic module of a McLain group into irreducible constituents, including multiplicities. Theorem 8.8 is a complete generalization of [A3, Theorem 2]. As in the case of $U_n(q)$, the decomposition of $V(D, f)$ is entirely controlled by that of $\text{ind}_H^I W$. Explicitly, our main result reads as follows.

Theorem 1.1. *Suppose R is finite and F is a splitting field for I over λ (in the sense of Definition 8.4). Then*

(a) *$\text{ind}_H^I W$ is a completely reducible I -module of finite length $\leq [I : H]$ and*

$$\text{ind}_H^I W = m_1 V_1 \oplus \cdots \oplus m_t V_t,$$

where $\{V_1, \dots, V_t\}$ is a full set of representatives for the isomorphism classes of irreducible I -modules lying over W (or, equivalently, λ). Moreover,

$$(W, \text{res}_H^I V_i)_H = \dim(V_i) = m_i, \quad 1 \leq i \leq t.$$

(b) *Assume, in addition, that Λ is well-ordered by \geq , the inverse order of \leq (this means: start with any well-order and impose its inverse on Λ). Then*

$$V(D, f) \cong m_1 \text{ind}_I^G V_1 \oplus \cdots \oplus m_t \text{ind}_I^G V_t,$$

where $\text{ind}_I^G V_1, \dots, \text{ind}_I^G V_t$ are non-isomorphic irreducible G -modules, and

$$(\text{ind}_I^G V_i, V(D, f))_M = \dim(V_i) = m_i, \quad 1 \leq i \leq t.$$

The requirement that \geq be a well-order is directly related to the complete reducibility issues mentioned above. Specifically, Theorem 8.7 shows that when R is finite and \geq is a well-order then I is a strongly ascendant subgroup of M (in the sense of Definition 6.4), which, in turn, ensures the complete reducibility of $V(D, f)$ by Theorem 6.5.

Some of the consequences of Theorem 1.1 appear to be unknown even for $U_n(q)$. Indeed, as in the case of $U_n(q)$, we have (cf. Theorem 7.5) that H is a normal subgroup of I , and I is included in the inertia group of W . What seems to be unknown is when the action of H on W is extendible to I . Theorem 8.14 answers this question: when D has no special triples, in the sense of Definition 8.12. Combining Theorems 8.8 with Theorem 8.14 and the generalization of Gallagher's theorem found in [Sz, Theorem 3.11], we obtain a much sharper decomposition of $V(D, f)$, as described in Theorem 8.15. This decomposition becomes even sharper if I/H is abelian, and this is stated in Corollary 8.16. Of course, the action of H on W is, in general, not extendible to I , and Example 8.19 illustrates how $V(D, f)$ decomposes in a family of such cases. Combining all of the results of this paragraph we obtain necessary and sufficient conditions for $V(D, f)$ to be multiplicity-free, as described in Theorem 8.21. This result also seems to be unknown for $U_n(q)$. Finally, Examples 8.19, 8.22 and 8.23 illustrate, in our context, André's result on basic characters which are multiplies of an irreducible character.

The main result of [A] is actually false for M . Indeed, it is shown in [Sz2] that when $\Lambda = \mathbb{Q}$ is ordered as usual and R is a division ring whose characteristic is not at the same time prime and equal to that of F , then M has a faithful irreducible module over F . But none of the basic modules for M are faithful for such Λ , so not every irreducible M -module is a constituent of a basic module.

2. McLAIN GROUPS

Let R be a ring with $1 \neq 0$, let Λ be a non-empty set partially ordered by \leq , and set

$$\Phi = \{(\alpha, \beta) \in \Lambda \times \Lambda \mid \alpha < \beta\}.$$

Let J be a free left R -module with basis $e_{\alpha\beta}$, $(\alpha, \beta) \in \Phi$. We define a multiplication on J by declaring

$$e_{\alpha\beta}e_{\beta\gamma} = e_{\alpha\gamma}, \quad e_{\alpha\beta}e_{\gamma\delta} = 0 \text{ if } \beta \neq \gamma,$$

and extending it to all of J by R -bilinearity. This makes J into a ring.

Lemma 2.1. *J is a nil ring.*

Proof. This follows by induction on the length of the longest chain in Φ present in an element of J when written as a linear combination of the $e_{\alpha\beta}$. \square

Adjoining an identity element to J , we obtain the group

$$M = M(\Lambda) = \{1 + x \mid x \in J\},$$

the *McLain group* associated to (Λ, \leq) over R . Every $g \in M$ has the form

$$(2.1) \quad g = 1 + \sum_{(\alpha, \beta) \in S(g)} r_{\alpha\beta} e_{\alpha\beta},$$

for a unique finite subset $S(g)$ of Φ and unique non-zero $r_{\alpha\beta} \in R$. Moreover, every such an element is in M . The following commutator formula is valid in M :

$$[1 + re_{\alpha\beta}, 1 + se_{\beta\gamma}] = 1 + rse_{\alpha\gamma}, \quad [1 + re_{\alpha\beta}, 1 + se_{\gamma\delta}] = 1 \text{ if } \beta \neq \gamma \text{ and } \alpha \neq \delta.$$

For $(\alpha, \beta) \in \Phi$, consider the subgroup $M_{\alpha\beta}$ of M defined by

$$M_{\alpha\beta} = \{1 + re_{\alpha\beta} \mid r \in R\}.$$

Clearly, the following map is a group isomorphism from R^+ onto $M_{\alpha\beta}$:

$$(2.2) \quad r \mapsto 1 + re_{\alpha\beta}, \quad r \in R^+.$$

For the remainder of this section we fix a subset Γ of Φ that is *closed*, in sense that $(\alpha, \beta), (\beta, \gamma) \in \Gamma$ implies $(\alpha, \gamma) \in \Gamma$. The *pattern* subgroup $M(\Gamma)$ of M corresponding to Γ is given by

$$M(\Gamma) = \langle M_{\alpha\beta} \mid (\alpha, \beta) \in \Gamma \rangle.$$

Suppose $\Gamma \subseteq \Gamma_1$, where Γ_1 is also a closed subset of Φ . We say that Γ is *normal* in Γ_1 if $(\alpha, \beta) \in \Gamma, (\beta, \gamma) \in \Gamma_1$ implies $(\alpha, \gamma) \in \Gamma$, and $(\alpha, \beta) \in \Gamma_1, (\beta, \gamma) \in \Gamma$ implies $(\alpha, \gamma) \in \Gamma$. Note that $M(\Gamma)$ is a normal subgroup of $M(\Gamma_1)$ if and only if Γ is a normal subset of Γ_1 .

We refer to Γ as *abelian* if Γ contains no chains. Clearly, $M(\Gamma)$ is an abelian subgroup of M if and only if Γ is an abelian subset of Φ .

Given $(\alpha, \beta) \in \Phi$, we consider

$$\begin{aligned} [\alpha, \beta] &= \{(\gamma, \delta) \in \Phi \mid \alpha \leq \gamma < \delta \leq \beta\}, \\ (\alpha, \leftarrow) &= \{(\gamma, \delta) \in \Phi \mid \gamma < \alpha\}, \quad (\alpha, \rightarrow) = \{(\gamma, \delta) \in \Phi \mid \alpha < \gamma\}, \\ (\rightarrow, \beta) &= \{(\gamma, \delta) \in \Phi \mid \beta < \delta\}, \quad (\leftarrow, \beta) = \{(\gamma, \delta) \in \Phi \mid \delta < \beta\}, \\ (\downarrow, \beta) &= \{(\gamma, \delta) \in \Phi \mid \delta = \beta\}, \quad [\alpha, \Rightarrow] = \{(\gamma, \delta) \in \Phi \mid \gamma = \alpha\}, \\ [\alpha, \downarrow, \beta] &= [\alpha, \beta] \cap (\downarrow, \beta), \\ [\alpha, \rightarrow, \beta] &= [\alpha, \beta] \cap [\alpha, \Rightarrow], \end{aligned}$$

as well as obvious variants of these obtained by interchanging closed and open brackets, in which case \leq and $<$ are to be interchanged.

At different stages of the paper we will use the basic but critical fact that $M(\Gamma)$ is actually a McLain group when Γ is any of the subsets $(\rightarrow, \beta], (\leftarrow, \beta), (\alpha, \beta)$.

All subsets of Γ displayed above are closed. Moreover, (α, \leftarrow) and (\rightarrow, β) are normal in Φ , while $[\alpha, \downarrow, \beta]$ and $[\alpha, \rightarrow, \beta]$ are abelian as well as normal in $[\alpha, \beta]$.

For the remainder of this section we suppose that \leq is a total order on Λ .

Definition 2.2. Consider the total order \preceq on Φ given by

$$(2.3) \quad (\alpha, \beta) \preceq (\gamma, \delta) \iff \beta < \delta, \text{ or } \beta = \delta \text{ and } \alpha \leq \gamma.$$

The elements of $M(\Gamma)$ can be uniquely expressed relative to this total order.

Proposition 2.3. Let $g \in M(\Gamma)$ be as in (2.1). Then

$$(2.4) \quad g = \prod_{(\alpha, \beta) \in S(g)} (1 + r_{\alpha\beta} e_{\alpha\beta}),$$

where the product is taken (from left to right) in decreasing \preceq -order.

Proof. This is trivial for $g = 1$. Assume $g \neq 1$ and suppose the distinct β 's that occur for $(\alpha, \beta) \in S(g)$ are $\beta_1 < \beta_2 < \dots < \beta_n$. Then

$$g = (1 + \sum_{\alpha} r_{\alpha\beta_n} e_{\alpha\beta_n}) (1 + \sum_{\beta < \beta_n} r_{\alpha\beta} e_{\alpha\beta}).$$

By induction on the size of $S(g)$, the last factor can be expressed as desired. Let $\alpha_1 < \alpha_2 < \dots < \alpha_m$ be the distinct α 's for which $(\alpha, \beta_n) \in S(g)$. Then

$$(1 + \sum_{\alpha} r_{\alpha\beta_n} e_{\alpha\beta_n}) = (1 + r_{\alpha_m\beta_n} e_{\alpha_m\beta_n}) (1 + r_{\alpha_{(m-1)}\beta_n} e_{\alpha_{(m-1)}\beta_n}) \cdots (1 + r_{\alpha_1\beta_n} e_{\alpha_1\beta_n}).$$

□

Note 2.4. Since (2.1) is uniquely determined by g then so is (2.4).

Proposition 2.5. Suppose $\Gamma \subseteq \Gamma_1$, where Γ_1 is a closed subset of Φ and Γ is normal in Γ_1 . Let $\Omega = \Gamma_1 \setminus \Gamma$ and set

$$T = \{1 + x \mid x \in \text{span}(e_{\alpha\beta})_{(\alpha,\beta) \in \Omega}\}.$$

Then T is transversal for $M(\Gamma)$ in $M(\Gamma_1)$.

Proof. By Proposition 2.3, T consists of all $g \in M(\Gamma_1)$ as in (2.4) with $S(g) \subseteq \Omega$. Let $y \in M(\Gamma_1)$. Since $M(\Gamma) \trianglelefteq M(\Gamma_1)$, Proposition 2.3 ensures the existence of $t \in T$ and $h \in M(\Gamma)$ such that $y = th$. Suppose $s, t \in T$ and $sM(\Gamma) = tM(\Gamma)$. We claim that $s = t$. Indeed, let

$$(2.5) \quad s = (1 + ae_{\alpha\beta})u \text{ and } t = (1 + be_{\alpha\beta})v$$

be the canonical expressions of s and t as elements of $M(\Gamma_1)$ ensured by Proposition 2.3, where $(\alpha, \beta) \in \Omega$ and u, v involve $(\gamma, \delta) \in \Omega$ such that $(\gamma, \delta) \prec (\alpha, \beta)$ as defined by (2.3). Possible trivial factors have been allowed so as to treat s and t simultaneously. Then

$$(2.6) \quad st^{-1} = (1 + ae_{\alpha\beta})(1 - be_{\alpha\beta})w,$$

where

$$w = (1 + be_{\alpha\beta})uv^{-1}(1 + be_{\alpha\beta})^{-1}.$$

Here uv^{-1} is a product of factors $1 + ce_{\gamma\delta}$ with $(\gamma, \delta) \prec (\alpha, \beta)$. In particular, $\gamma < \beta$, so either $1 + be_{\alpha\beta}$ and $1 + ce_{\gamma\delta}$ commute or $\gamma < \delta = \alpha$, in which case

$$(1 + be_{\alpha\beta})(1 + ce_{\gamma\alpha})(1 + be_{\alpha\beta})^{-1} = (1 + ce_{\gamma\alpha})(1 - cbe_{\gamma\beta}).$$

Thus $w = 1 + x$, where x is an R -linear combination of $e_{\gamma\delta}$ with $(\gamma, \delta) \prec (\alpha, \beta)$. Therefore by (2.6),

$$st^{-1} = (1 + (a - b)e_{\alpha\beta})(1 + x) = 1 + (a - b)e_{\alpha\beta} + x.$$

On the other hand, since $M(\Gamma)s = M(\Gamma)t$ and $\Omega \cap \Gamma = \emptyset$, the coefficient of $e_{\alpha\beta}$ when $st^{-1} \in M(\Gamma)$ is written as in (2.1) must be 0. This shows $a = b$. Now cancel the leftmost factors of s and t in (2.5) and repeat. □

Definition 2.6. Given a subset Δ of Φ we define $[\Delta, \Delta]$ to be the closed subset of Φ consisting of all (α, β) such that there is a chain $(\gamma_1, \gamma_2), \dots, (\gamma_{n-1}, \gamma_n) \in \Delta$, $n \geq 3$, satisfying $\gamma_1 = \alpha$ and $\gamma_n = \beta$.

Proposition 2.7. (a) $[M(\Gamma), M(\Gamma)] = M([\Gamma, \Gamma])$.

(b) Let $\rho : M(\Gamma) \rightarrow M(\Gamma)/[M(\Gamma), M(\Gamma)]$ be the canonical projection, and consider the map

$$\Theta : \prod_{(\alpha, \beta) \in \Gamma \setminus [\Gamma, \Gamma]} M_{\alpha\beta} \rightarrow M(\Gamma)/[M(\Gamma), M(\Gamma)],$$

where the left hand side is the external direct product of the given $M_{\alpha\beta}$, and Θ is defined by

$$\prod_{(\alpha, \beta) \in \Gamma \setminus [\Gamma, \Gamma]} (1 + r_{\alpha\beta} e_{\alpha\beta}) \rightarrow \rho \left(\prod_{(\alpha, \beta) \in \Gamma \setminus [\Gamma, \Gamma]} (1 + r_{\alpha\beta} e_{\alpha\beta}) \right).$$

Here almost all $r_{\alpha\beta}$ are 0 and the product on the right hand side is taken in decreasing order under \preceq (the order on the left hand side is obviously irrelevant). Then Θ is a group isomorphism.

Proof. (a) By definition, $M([\Gamma, \Gamma])$ is generated by all $M_{\sigma\tau}$ with $(\sigma, \tau) \in [\Gamma, \Gamma]$. Given such (σ, τ) there are $(\alpha_1, \alpha_2), \dots, (\alpha_{n-1}, \alpha_n) \in \Gamma$, $n \geq 3$, such that $\alpha_1 = \sigma$ and $\alpha_n = \tau$. A repeated application of the commutator formula shows that $M_{\sigma\tau}$ is contained in $[M(\Gamma), M(\Gamma)]$, and therefore $M([\Gamma, \Gamma]) \subseteq [M(\Gamma), M(\Gamma)]$.

On the other hand, $M(\Gamma)$ is generated by all $M_{\alpha\beta}$, $(\alpha, \beta) \in \Gamma$. Thus, by the commutator formula, $M(\Gamma)/M([\Gamma, \Gamma])$ is abelian, whence $[M(\Gamma), M(\Gamma)] \subseteq M([\Gamma, \Gamma])$.

(b) This follows from Proposition 2.3 as well as from Proposition 2.5. \square

3. CLIFFORD THEORY FOR INFINITE DIMENSIONAL MODULES

We fix a field F for the remainder of the paper. Let $N \trianglelefteq G$ be groups and let W be an N -module. For $g \in G$, consider the N -module W^g , whose underlying F -vector space is W , acted upon by N as follows:

$$x \cdot w = (gxg^{-1})w, \quad x \in N, w \in W.$$

Then

$$I_G(W) = \{g \in G \mid W^g \cong W\}$$

is a subgroup of G called the inertia group of W (cf. [Sz, Lemma 3.1]).

Theorem 3.1. Let $N \trianglelefteq G$ be groups and let W be an irreducible N -module with inertia group T . Then $S \mapsto \text{ind}_T^G S$ yields a bijective correspondence between the isomorphism classes of irreducible modules of T and G lying over W .

In particular, if $I_G(W) = N$ then $V = \text{ind}_N^G W$ is irreducible and if, in addition, $\text{End}_N(W) = F$ then $\text{End}_G(V) = F$ as well.

Proof. See [Sz, Theorem 3.5] for the first assertion. As for the second, by Frobenius reciprocity (cf. [Sz, Theorem 5.3]), $\text{End}_N(W) = \text{Hom}_N(W, V) \cong_F \text{End}_G(V)$. \square

Lemma 3.2. Let $N \trianglelefteq G$ be groups and let W be an irreducible G -module such that $\text{res}_N^G W$ remains irreducible and $\text{End}_N(W) = F$. Let U_1, U_2 be G -modules acted upon trivially by N and suppose $T \in \text{Hom}_G(U_1 \otimes W, U_2 \otimes W)$. Then $T = S \otimes 1$, where $S \in \text{Hom}_{G/N}(U_1, U_2)$.

Proof. Since $T \in \text{Hom}_N(U_1 \otimes W, U_2 \otimes W)$, [Sz, Lemma 3.7] implies $T = S \otimes 1$, where $S \in \text{Hom}_F(U_1, U_2)$. But $T \in \text{Hom}_G(U_1 \otimes W, U_2 \otimes W)$, so $S \in \text{Hom}_{G/N}(U_1, U_2)$. \square

Corollary 3.3. *Let $N \trianglelefteq G$ be groups and let W be a G -module such that $\text{res}_N^G W$ is a multiplicity-free, completely reducible N -module, such that $\text{End}_N(X) = F$ for every irreducible constituent X of $\text{res}_N^G W$. Let U_1, U_2 be G -modules acted upon trivially by N and suppose $T \in \text{Hom}_G(U_1 \otimes W, U_2 \otimes W)$. Then $T = S \otimes 1$, where $S \in \text{Hom}_{G/N}(U_1, U_2)$.*

Theorem 3.4. *Let $N \trianglelefteq G$ be groups and let W be an irreducible G -module with $\text{res}_N^G W$ irreducible and $\text{End}_N(W) = F$. Then $U \mapsto U \otimes W$ yields a bijective correspondence between the isomorphism classes of irreducible G -modules acted upon trivially by N and irreducible G -modules lying over W .*

Proof. This can be found in [Sz, Theorem 3.11]. \square

Theorem 3.5. *Let G be a group with subgroups H_1, H_2 and irreducible G -modules V_1, V_2 such that: H_1 acts trivially on V_2 ; H_2 acts trivially on V_1 ; V_i is irreducible as H_i -module and $\text{End}_{H_i}(V_i) = F$ for $1 \leq i \leq 2$. Then $V = V_1 \otimes V_2$ is an irreducible module for $H = \langle H_1, H_2 \rangle$ and $\text{End}_H(V) = F$.*

Proof. See the proof of [Sz, Theorem 3.10] for the first assertion. The second follows from [Sz, Lemma 3.7]. \square

Corollary 3.6. *Let G be a group with subgroups H_1, \dots, H_n and irreducible G -modules V_1, \dots, V_n such that: H_i acts trivially on each V_j , $j \neq i$; V_i is irreducible as H_i -module and $\text{End}_{H_i}(V_i) = F$ for $1 \leq i \leq n$. Then $V = V_1 \otimes \dots \otimes V_n$ is an irreducible G -module.*

Proof. This follows from Theorem 3.5 by induction. \square

4. ELEMENTARY MODULES

Definition 4.1. A linear character $\lambda : R^+ \rightarrow F^*$ is said to be *right* (resp. *left*) *primitive* if the only right (resp. left) ideal of R contained in the kernel of λ is (0) .

We assume for the remainder of the paper that R admits a right primitive linear character $\lambda : R^+ \rightarrow F^*$.

Lemma 4.2. *Suppose F has prime characteristic p . Then R^+ has no element of order p .*

Proof. The set $\{r \in R \mid p \cdot r = 0\}$ is an ideal of R contained in the kernel of every group homomorphism $R^+ \rightarrow F^*$. \square

Lemma 4.3. *Suppose R^+ has finite exponent. Then F^* has a root of unity of order $\exp(R^+)$.*

Proof. The prime factorization $\exp(R^+) = p_1^{a_1} \dots p_n^{a_n}$ yields the factorization $R^+ = R_1 \oplus \dots \oplus R_n$, as rings, where $R_i = \{r \in R \mid p_i^{a_i} \cdot r = 0\}$. Now $(0) \neq p_i^{a_i-1} R_i \trianglelefteq R$, so for each i there is $r_i \in R_i$ such that $\lambda(p_i^{a_i-1} r_i) \neq 1$. But $\lambda(p_i^{a_i} r_i) = 1$, so each $\lambda(r_i)$ has order $p_i^{a_i}$, whence $\lambda(r_1) \dots \lambda(r_n)$ has order $\exp(R^+)$. \square

Given $(\alpha, \beta) \in \Phi$, consider the normal subgroup $M^{\alpha\beta}$ of $M([\alpha, \beta])$ defined by

$$M^{\alpha\beta} = M([\alpha, \downarrow, \beta])M([\alpha, \rightarrow, \beta]).$$

Let $W = Fw$ be a 1-dimensional $M_{\alpha\beta}$ -module upon which $M_{\alpha\beta}$ acts, through (2.2), by means of λ .

Note that $M([\alpha, \downarrow, \beta])$ is the direct product of $M((\alpha, \downarrow, \beta))$ and $M_{\alpha\beta}$, so we can extend λ to a linear character λ' of $M([\alpha, \downarrow, \beta])$ that is trivial on $M((\alpha, \downarrow, \beta))$. We make $M([\alpha, \downarrow, \beta])$ act on W via λ' .

Now $M([\alpha, \downarrow, \beta])$ is normal in $M^{\alpha\beta}$, and the inertia group of the $M([\alpha, \downarrow, \beta])$ -module W in $M^{\alpha\beta}$ is $M([\alpha, \downarrow, \beta])$. The latter holds because λ is right primitive and R has 1. It follows from Theorem 3.1 that

$$U(\alpha, \beta, \lambda) = \text{ind}_{M([\alpha, \downarrow, \beta])}^{M^{\alpha\beta}} W$$

is an irreducible $M^{\alpha\beta}$ -module satisfying $\text{End}_{M^{\alpha\beta}} U(\alpha, \beta, \lambda) = F$. Moreover, if the open interval (α, β) of Λ is finite, it is easy to see that

$$\dim(U(\alpha, \beta, \lambda)) = |R|^{|\langle \alpha, \beta \rangle|}.$$

On the other hand, $M([\alpha, \downarrow, \beta])$ is normal in $M([\alpha, \beta])$ and the inertia group of the $M([\alpha, \downarrow, \beta])$ -module W in $M([\alpha, \beta])$ is $M([\alpha, \downarrow, \beta]) \rtimes M((\alpha, \beta))$. As W is 1-dimensional, the action of $M([\alpha, \downarrow, \beta])$ on W is extendible to $M([\alpha, \downarrow, \beta]) \rtimes M((\alpha, \beta))$ by letting $M((\alpha, \beta))$ act trivially on W . It follows from Theorem 3.1 that

$$V(\alpha, \beta, \lambda) = \text{ind}_{M([\alpha, \downarrow, \beta]) \rtimes M((\alpha, \beta))}^{M([\alpha, \beta])} W$$

is an irreducible $M([\alpha, \beta])$ -module, known as *elementary*. But

$$M^{\alpha\beta} = M([\alpha, \downarrow, \beta]) \rtimes M([\alpha, \rightarrow, \beta])$$

and

$$M([\alpha, \beta]) = (M([\alpha, \downarrow, \beta]) \rtimes M([\alpha, \rightarrow, \beta])) \rtimes M((\alpha, \beta))$$

so

$$\text{res}_{M^{\alpha\beta}}^{M([\alpha, \beta])} V(\alpha, \beta, \lambda) \cong U(\alpha, \beta, \lambda).$$

We have shown

Theorem 4.4. *Let $(\alpha, \beta) \in \Phi$ and suppose $\lambda : R^+ \rightarrow F^*$ is primitive. Then $V(\alpha, \beta, \lambda)$ is an irreducible $M([\alpha, \beta])$ -module, which remains irreducible as an $M^{\alpha\beta}$ -module and satisfies $\text{End}_{M^{\alpha\beta}} V(\alpha, \beta, \lambda) = F$.*

Next we address the classification of irreducible $M^{\alpha\beta}$ -modules lying over λ .

Theorem 4.5. *Suppose R is finite and the interval (α, β) of Λ is also finite. Then $U(\alpha, \beta, \lambda)$ is the only irreducible $M^{\alpha\beta}$ -module, up to isomorphism, lying over λ .*

Proof. By Lemma 4.2 and Maschke's theorem, every $M^{\alpha\beta}$ -module is completely reducible. By construction, W enters $|R|^{|\langle \alpha, \beta \rangle|}$ times in $U(\alpha, \beta, \lambda)$, so $U(\alpha, \beta, \lambda)$ enters $|R|^{|\langle \alpha, \beta \rangle|}$ times in $\text{ind}_{M^{\alpha\beta}}^{M^{\alpha\beta}} W$ by Frobenius reciprocity. But $|R|^{|\langle \alpha, \beta \rangle|} U(\alpha, \beta, \lambda)$ and $\text{ind}_{M^{\alpha\beta}}^{M^{\alpha\beta}} W$ have the same degree, so the result follows. \square

Corollary 4.6. *Suppose R is finite. Then λ is also left primitive.*

Proof. Let K be the largest left ideal of R contained in the kernel of λ . By Theorem 4.5 there is one and only one irreducible module of $M = U_3(R)$, up to isomorphism, lying over λ , viewed as a linear character of M_{13} . Let $S = M_{12}M_{13} \trianglelefteq M$ and extend λ to $\mu : S \rightarrow F^*$ so that μ is trivial on M_{12} . The inertia group of μ is $S \rtimes T$, where T consists of all $1 + re_{23}$ with $r \in K$. Since T stabilizes μ , we can extend μ to $\nu : S \rtimes T \rightarrow F^*$, so that ν is trivial on T . Now $S \rtimes T$ is normal in M and the inertia group of ν is $S \rtimes T$. Induction produces an irreducible M -module of degree $[R : K]$ lying over λ . By uniqueness, $K = (0)$. \square

Naturally, we could have started with the assumption that λ is left primitive to deduce that λ is right primitive. We thus obtain, through representation theory, an answer to the question posed by Claassen and Goldbach [CG, §8], already answered by Wood [Wo, Theorem 4.3] by a different method.

Theorem 4.7. *Suppose the interval (α, β) is infinite. Then there are uncountably many non-isomorphic irreducible $M^{\alpha\beta}$ -modules lying over λ . All are completely reducible upon restriction to $M([\alpha, \downarrow, \beta])$, none is completely reducible upon restriction to $M([\alpha, \rightarrow, \beta])$, and therefore none has irreducible submodules upon restriction to the normal subgroup $M([\alpha, \rightarrow, \beta])$.*

Proof. Given a family $(\mu_\gamma)_{\gamma \in (\alpha, \beta)}$ of linear characters $R \rightarrow F^*$, we extend λ to a linear character $\mu : M([\alpha, \downarrow, \beta]) \rightarrow F^*$ that agrees with μ_γ on $M_{\gamma\beta}$ for all γ in (α, β) . The inertia group of μ is still $M([\alpha, \downarrow, \beta])$. Inducing up to $M^{\alpha\beta}$ we obtain an irreducible $M^{\alpha\beta}$ -module U_μ lying over λ . A conjugate of μ by an element of $M([\alpha, \rightarrow, \beta])$ agrees with μ on $M_{\gamma\beta}$ for all but finitely many $\gamma \in (\alpha, \beta)$. On the other hand, by Clifford's theorem, if U is an irreducible $M^{\alpha\beta}$ -module, then $U \cong U_\mu$ if and only if U lies over a $M([\alpha, \rightarrow, \beta])$ -conjugate of μ .

Consider the equivalence relation on the set X of infinite subsets of \mathbb{N} , given by $A \sim B$ if $A \Delta B$ (symmetric difference of A and B) is finite. Each equivalence class has countably many elements and X is uncountable, so the number of equivalence classes is uncountable. Since (α, β) is infinite, we arrive at the same conclusion if we replace \mathbb{N} by (α, β) . It follows from above that, by varying μ , we obtain uncountably many non-isomorphic irreducible $M^{\alpha\beta}$ -modules that lie over λ and are completely reducible upon restriction to $M([\alpha, \downarrow, \beta])$.

As $M_{\alpha\beta}$ acts via scalar operators on U_μ , our claim for $\text{res}_{M([\alpha, \rightarrow, \beta])}^{M^{\alpha\beta}} U_\mu$ is equivalent to that for $\text{res}_{M([\alpha, \rightarrow, \beta])}^{M^{\alpha\beta}} U_\mu$. Restriction to $M([\alpha, \rightarrow, \beta])$ yields the regular $M([\alpha, \rightarrow, \beta])$ -module, say P . The epimorphism $P \rightarrow F$ shows that a supposed complement to the augmentation ideal must be trivial. But P has no trivial submodule because (α, β) , and hence $M([\alpha, \rightarrow, \beta])$, is infinite. Thus $\text{res}_{M([\alpha, \rightarrow, \beta])}^{M^{\alpha\beta}} U_\mu$ is not completely reducible, and therefore lacks irreducible $M([\alpha, \rightarrow, \beta])$ -submodules, by Clifford's theorem. \square

Note 4.8. When $\exp(R^+)$ is finite, Theorem 4.7 gives an example of an infinite family of diagonalizable linear operators acting on an infinite dimensional vector space that are simultaneously diagonalizable and which, in fact, possess no common eigenvector. If either the space or the span of the family are finite dimensional, this phenomenon is impossible. The family simply consists of the operators corresponding to each $\gamma \in [\alpha, \rightarrow, \beta)$ acting on U_μ . They commute because $[\alpha, \rightarrow, \beta)$ is abelian and they are individually diagonalizable because they are all annihilated by the polynomial $t^{\exp(R^+)} - 1 \in F[t]$, which splits completely over F by Lemma 4.3.

Next we show that $M^{\alpha\beta}$ has yet another uncountable family of non-isomorphic irreducible modules, none of which is isomorphic to any of the modules constructed in Theorem 4.7.

Theorem 4.9. *Suppose λ is also left primitive (this is automatic if R is finite, by Lemma 4.6, and also if R is commutative) and the interval (α, β) is infinite. Then there are uncountably many non-isomorphic irreducible $M^{\alpha\beta}$ -modules lying*

over λ . All are completely reducible upon restriction to $M([\alpha, \rightarrow, \beta])$, none is completely reducible upon restriction to $M([\alpha, \downarrow, \beta])$, and therefore none has irreducible submodules upon restriction to the normal subgroup $M([\alpha, \downarrow, \beta])$.

Proof. Reason as above. First extend λ to $[\alpha, \rightarrow, \beta]$ and then induce up to $M^{\alpha\beta}$. \square

5. BASIC MODULES

We assume for the remainder of the paper that \leq is a total order on Λ . Then

$$(5.1) \quad M = M(\alpha, \leftarrow)M(\rightarrow, \beta) \rtimes M([\alpha, \beta]),$$

so that each elementary module $V(\alpha, \beta, \lambda)$ can be viewed as an M -module acted upon trivially by $M(\alpha, \leftarrow)M(\rightarrow, \beta)$.

Definition 5.1. A subset $D = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ of Φ is said to be *basic* if all α_i (resp. all β_i) are distinct.

Given a basic subset D of Φ and a function f from D into the set of all right primitive linear characters $R^+ \rightarrow F^*$, say $f(\alpha_i, \beta_i) = \lambda_i$, $1 \leq i \leq m$, the *basic* M -module $V(D, f)$ associated to (D, f) is given by

$$V(D, f) = V(\alpha_m, \beta_m, \lambda_m) \otimes \cdots \otimes V(\alpha_1, \beta_1, \lambda_1).$$

Theorem 5.2. Suppose $U = V(D, f)$ and $U' = V(D', f')$ are basic M -modules satisfying

$$\text{Hom}_M(U, U') \neq 0.$$

Then $D = D'$ and $f = f'$.

Proof. Relabelling, if necessary, we may assume that β_1 (resp. β'_1) is the largest element of $\{\beta_1, \dots, \beta_m\}$ (resp. $\{\beta'_1, \dots, \beta'_{m'}\}$).

By construction, each $V(\alpha_i, \beta_i, \lambda_i)$ (resp. $V(\alpha'_i, \beta'_i, \lambda'_i)$), $i > 1$, is acted upon trivially by $M_{\alpha_1\beta_1}$ (resp. $M_{\alpha'_1\beta'_1}$). Therefore $M_{\alpha_1\beta_1}$ (resp. $M_{\alpha'_1\beta'_1}$) acts on U (resp. U') via scalar operators determined by λ_1 (resp. λ'_1).

Suppose, if possible, that $\beta_1 < \beta'_1$ (resp. $\beta'_1 < \beta_1$). Then $M_{\alpha'_1\beta'_1}$ (resp. $M_{\alpha_1\beta_1}$) acts trivially on U (resp. U'). Thus

$$\text{Hom}_{M_{\alpha'_1\beta'_1}}(U, U') = 0 \text{ (resp. } \text{Hom}_{M_{\alpha_1\beta_1}}(U, U') = 0),$$

whence $\text{Hom}_M(U, U') = 0$, a contradiction. This forces $\beta_1 = \beta'_1$.

Suppose, if possible, that $\alpha'_1 < \alpha_1$ (resp. $\alpha_1 < \alpha'_1$). Then $M_{\alpha'_1\beta'_1}$ (resp. $M_{\alpha_1\beta_1}$) acts trivially on U (resp. U'), regardless of how α_1 and α'_1 compare to the other α_i and α'_i . As above, this leads to the contradiction $\text{Hom}_M(U, U') = 0$, so $\alpha_1 = \alpha'_1$.

Since $M_{\alpha_1\beta_1}$ acts on U (resp. U') via scalar operators determined by λ_1 (resp. λ'_1), the condition $\text{Hom}_M(U, U') \neq 0$ forces $\lambda_1 = \lambda'_1$.

On the other hand, we have the decomposition

$$M = M((\rightarrow, \beta_1)) \rtimes M((\leftarrow, \beta_1]),$$

where $M((\rightarrow, \beta_1))$ acts trivially on U and U' , so

$$(5.2) \quad \text{Hom}_{M((\leftarrow, \beta_1])}(U, U') \neq 0.$$

Set

$$W = V(\alpha_m, \beta_m, \lambda_m) \otimes \cdots \otimes V(\alpha_2, \beta_2, \lambda_2), \quad W' = V(\alpha'_{m'}, \beta'_{m'}, \lambda'_{m'}) \otimes \cdots \otimes V(\alpha'_2, \beta'_2, \lambda'_2),$$

with W (resp. W') the trivial module if $m = 1$ (resp. $m' = 1$). We also have the decomposition

$$(5.3) \quad M((\leftarrow, \beta_1]) = M((\downarrow, \beta_1]) \rtimes M((\leftarrow, \beta_1)),$$

where $M((\downarrow, \beta_1])$ is a normal subgroup of $M((\leftarrow, \beta_1])$ acting trivially on W and W' . Moreover, by above,

$$U = W \otimes V(\alpha_1, \beta_1, \lambda_1) \text{ and } U' = W' \otimes V(\alpha_1, \beta_1, \lambda_1).$$

We readily see that $\text{res}_{M((\downarrow, \beta_1])}^{M((\leftarrow, \beta_1])} V(\alpha_1, \beta_1, \lambda_1)$ is a multiplicity-free completely reducible $M((\downarrow, \beta_1])$ -module with 1-dimensional irreducible constituents. It follows from Corollary 3.3, (5.2) and (5.3) that

$$\text{Hom}_{M((\leftarrow, \beta_1])}(W, W') \neq 0,$$

where $M((\leftarrow, \beta_1])$ is a McLain group. The above argument shows that $m = 1$ if and only if $m' = 1$, while otherwise W and W' are basic $M((\leftarrow, \beta_1])$ -modules, and induction applies. \square

Notation 5.3. We fix a basic subset $D = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ of Φ for the remainder of the paper, as well as a function f from D into the set of all right primitive linear characters $R^+ \rightarrow F^*$, say $f(\alpha_i, \beta_i) = \lambda_i$, $1 \leq i \leq m$. Moreover, in the presence of (D, f) , the following notation will be in effect from now on:

$$\begin{aligned} \Gamma &= \bigcap_{1 \leq i \leq m} (\Phi \setminus [\alpha_i, \rightarrow, \beta_i)) = \Phi \setminus \left(\bigcup_{1 \leq i \leq m} [\alpha_i, \rightarrow, \beta_i) \right), \\ H_i &= M(\Phi \setminus [\alpha_i, \rightarrow, \beta_i)), \quad 1 \leq i \leq m, \\ H &= M(\Gamma) = \bigcap_{1 \leq i \leq m} H_i, \\ \widehat{\lambda}_i &: H_i \rightarrow F^*, \end{aligned}$$

the group homomorphism that extends $\lambda_i : M_{\alpha_i \beta_i} \rightarrow F^*$ (we are identifying $M_{\alpha_i \beta_i}$ with R^+ via (2.2)) and is trivial on all $M_{\alpha \beta}$ with $(\alpha, \beta) \in \Phi \setminus [\alpha_i, \rightarrow, \beta_i]$,

$$\lambda = \widehat{\lambda}_1|_H \cdots \widehat{\lambda}_m|_H,$$

$$W = Fw,$$

a 1-dimensional H -module acted upon by H via λ .

Theorem 5.4. $V(D, f) \cong \text{ind}_H^M W$.

Proof. We have $V(D, f) = Y \otimes V(\alpha_1, \beta_1, \lambda_1)$, where

$$Y = V(\alpha_m, \beta_m, \lambda_m) \otimes \cdots \otimes V(\alpha_2, \beta_2, \lambda_2), \quad V(\alpha_1, \beta_1, \lambda_1) = \text{ind}_{H_1}^M Z$$

and $Z = Fz$ is acted upon H_1 via $\widehat{\lambda}_1$. By induction, $Y \cong \text{ind}_{H_0}^M T$, where $H_0 = H_2 \cap \cdots \cap H_m$ and $T = Ft$ is acted upon H_0 via $\widehat{\lambda}_2|_{H_0} \cdots \widehat{\lambda}_m|_{H_0}$. Since $H_0 H_1 = M$ and $H_0 \cap H_1 = H$, Mackey Tensor Product Theorem (cf. [Sz, Theorem 2.1]) yields $V(D, f) \cong \text{ind}_H^M W$. \square

Definition 5.5. Let $\alpha < \beta$ and $\gamma < \delta$ be in Λ . We say that the intervals (α, β) and (γ, δ) of Λ are *nested* if $\alpha < \gamma < \delta < \beta$ or $\gamma < \alpha < \beta < \delta$, and *overlapping* if $\alpha < \gamma < \beta < \delta$ or $\gamma < \alpha < \delta < \beta$.

Definition 5.6. We say that D is *nested* (resp. *non-overlapping*) if the open intervals (resp. none of the open intervals) of Λ determined by the elements of D are nested (resp. overlapping).

Definition 5.7. We say that a disjoint union $D = D_1 \cup \dots \cup D_n$ is a *disconnection* if each D_i is non-empty and, for each $1 \leq i < n$, the conditions $(\alpha, \beta) \in D_i$ and $(\gamma, \delta) \in D_{i+1}$ imply $\beta \leq \gamma$.

Notation 5.8. Given a group G and G -modules U and V , we set

$$(U, V)_G = \dim_F \text{Hom}_G(U, V).$$

Theorem 5.9. Suppose D is nested. Then $V(D, f)$ is irreducible and

$$(V(D, f), V(D, f))_M = 1.$$

Proof. The fact that D is nested translates as follows:

$$\alpha_1 < \alpha_2 < \dots < \alpha_m < \beta_m < \dots < \beta_2 < \beta_1.$$

By Theorem 4.4, $V(\alpha_1, \beta_1, \lambda_1)$ is an irreducible M -module whose restriction to $M^{\alpha_1 \beta_1}$ remains irreducible and $\text{End}_{M^{\alpha_1 \beta_1}} V(\alpha_1, \beta_1, \lambda_1) = F$. By Lemma 3.2 and Theorem 3.4, if W is any irreducible M -module acted upon trivially by $M^{\alpha_1 \beta_1}$ then $W \otimes V(\alpha_1, \beta_1, \lambda_1)$ is an irreducible M -module and

$$(W \otimes V(\alpha_1, \beta_1, \lambda_1), W \otimes V(\alpha_1, \beta_1, \lambda_1))_M = (W, W)_M.$$

We have the decomposition

$$M = M([\alpha_1, \leftarrow))M((\rightarrow, \beta_1]) \rtimes M((\alpha_1, \beta_1)),$$

where $M([\alpha_1, \leftarrow))M((\rightarrow, \beta_1])$, and hence $M^{\alpha_1 \beta_1}$, acts trivially on each $V(\alpha_i, \beta_i, \lambda_i)$, $1 < i \leq m$. Moreover, $W = V(\alpha_m, \beta_m, \lambda_m) \otimes \dots \otimes V(\alpha_2, \beta_2, \lambda_2)$ is a nested module of the McLain group $M((\alpha_1, \beta_1))$. By induction, W is an irreducible $M((\alpha_1, \beta_1))$ -module, where $(W, W)_{M((\alpha_1, \beta_1))} = 1$, and therefore W is an irreducible M -module acted upon trivially by $M^{\alpha_1 \beta_1}$ and satisfying $(W, W)_M = 1$. The result follows. \square

Theorem 5.10. Suppose D is non-overlapping. Then $V(D, f)$ is irreducible.

Proof. By hypothesis D has a disconnection $D = D_1 \cup \dots \cup D_n$, where each D_i is nested. Let (α_i, β_i) be the outermost interval of D_i , so that $\beta_i \leq \alpha_{i+1}$ for each $1 \leq i < n$. Let f_i be the restriction of f to each D_i . By Theorem 5.9, each $V(D_i, f_i)$ is an irreducible $M([\alpha_i, \beta_i])$ -module satisfying $(V(D_i, f_i), V(D_i, f_i))_{M([\alpha_i, \beta_i])} = 1$, so $V(D, f)$ is an irreducible M -module by Corollary 3.6. \square

Lemma 5.11. Let $H \leq G$ be groups and let W be an H -module. Let I be a subgroup of G properly containing H . Suppose the action of H on W is extendible to I and call the resulting I -module W_1 . Then $\text{ind}_H^G W$ is reducible.

Proof. Let $T = Ft$ be the trivial H -module and let P be the permutation I -module associated to the coset space I/H . Then

$$\begin{aligned} \text{ind}_H^G W &\cong \text{ind}_I^G \text{ind}_H^I W \\ &\cong \text{ind}_I^G \text{ind}_H^I (T \otimes W) \\ &\cong \text{ind}_I^G \text{ind}_H^I (T \otimes \text{res}_H^I W_1) \\ &\cong \text{ind}_I^G ((\text{ind}_H^I T) \otimes W_1) \quad (\text{cf. [Sz, §2]}) \\ &\cong \text{ind}_I^G (P \otimes W_1). \end{aligned}$$

Since a permutation module of dimension > 1 (finite or infinite) is always reducible, so is $\text{ind}_H^G W$. \square

In regards to Lemma 5.11, see the examples given in [Sz, §5].

Theorem 5.12. *Suppose D is overlapping. Then $V(D, f)$ is reducible.*

Proof. It suffices to prove the result when $|D| = 2$. We have $V(D, f) \cong \text{ind}_H^M W$, as in Theorem 5.4.

Note that $M_{\alpha_1\alpha_2} \cap H = 1$. Moreover, $M_{\alpha_1\alpha_2}$ normalizes H and stabilizes λ . Therefore λ is extendible to $I = H \rtimes M_{\alpha_1\alpha_2}$ via any map

$$h(1 + re_{\alpha_1\alpha_2}) \mapsto \lambda(h)\chi(r)$$

with $\chi \in \text{Hom}(R^+, F^*)$. Now apply Lemma 5.11. \square

Theorem 5.13. *$V(D, f)$ is irreducible if and only if D is non-overlapping.*

Proof. Immediate consequence of Theorems 5.10 and 5.12. \square

6. TOOLS REQUIRED TO STUDY THE DECOMPOSITION OF A BASIC MODULE

Theorem 6.1. *Let $H \trianglelefteq I \leq G$ be groups and let W be an irreducible H -module stabilized by I . Given any $x \in G$ let ${}^x W$ be the vector space W when viewed as an $H \cap xHx^{-1}$ -module via*

$$h \cdot w = (x^{-1}hx)w.$$

Suppose that

$$(W, W)_H = 1$$

and that given any $x \in G \setminus I$, we have

$$(W, {}^x W)_{H \cap xHx^{-1}} = 0.$$

Then

$$(\text{ind}_H^I W, \text{ind}_H^I W)_I = [I : H] = (\text{ind}_H^G W, \text{ind}_H^G W)_G.$$

Note 6.2. The first equality is trivial. Under more restrictive hypothesis the second equality would be a routine application of Frobenius reciprocity, twice, with Mackey Decomposition Theorem used in between. However, $(X, Y)_S$ need not equal $(Y, X)_S$, in general, and this property would be required twice in that argument, first for $S = H$ and then for $S = H \cap xHx^{-1}$. The second use could be avoided if we modified our hypotheses, but the first could not.

The alternative argument presented below, which goes inside of the proof of the Mackey Decomposition Theorem, will suffice for our purposes.

Proof. Since $H \trianglelefteq I$ and I stabilizes W , we see that $\text{res}_H^I \text{ind}_H^I W$ is the direct sum of $[I : H]$ copies of W , so by Frobenius reciprocity

$$(\text{ind}_H^I W, \text{ind}_H^I W)_I = (W, \text{res}_H^I \text{ind}_H^I W)_H = [I : H].$$

Let E be a system of representatives for the (H, H) -double cosets in G . Since $H \trianglelefteq I$, we may assume that E contains a system, say E_0 , of representatives for I/H . Moreover, for each $x \in E$, let T_x be a system of representatives, including 1, for the left cosets of $H \cap xHx^{-1}$ in H . Then

$$S = \{yx \mid x \in E, y \in T_x\}$$

is a set of representatives for the left cosets of H in G . From

$$\operatorname{ind}_H^G W = \bigoplus_{g \in S} gW,$$

we obtain

$$\operatorname{res}_H^G \operatorname{ind}_H^G W = \bigoplus_{x \in E} U_x,$$

where U_x is the H -submodule of $\operatorname{ind}_H^G W$ given by

$$U_x = \bigoplus_{y \in T_x} yxW.$$

Suppose first $x \in E_0$. Then $T_x = \{1\}$ and $U_x = xW \cong W$, so

$$(W, U_x)_H = 1.$$

Suppose next $x \in E \setminus E_0$. Then $yx \notin I$ for any $y \in T_x$. Therefore

$$(W, yxW)_{H \cap yxH(yx)^{-1}} = 0, \quad y \in T_x,$$

and fortiori

$$(W, U_x)_H = 0.$$

Since W is H -irreducible, and hence generated by a single element as H -module, it follows that $\operatorname{Hom}_H(W, -)$ commutes with the direct sum of H -modules. Thus, by above

$$(W, \operatorname{res}_H^G \operatorname{ind}_H^G W)_H = [I : H],$$

so, by Frobenius reciprocity,

$$(\operatorname{ind}_H^G W, \operatorname{ind}_H^G W)_G = [I : H].$$

□

Proposition 6.3. *Let $S \trianglelefteq T$ be groups, where $[T : S]$ is finite and $\operatorname{char}(F) \nmid [T : S]$. Let W be a completely reducible S -module. Then $\operatorname{ind}_S^T W$ is a completely reducible T -module.*

Proof. We have $W = \bigoplus_{x \in X} W_x$ with W_x irreducible, so $\operatorname{ind}_S^T W \cong \bigoplus_{x \in X} \operatorname{ind}_S^T W_x$. We may thus restrict to the case when W itself is irreducible. Since $\operatorname{res}_S^T \operatorname{ind}_S^T W$ is the direct sum of conjugates of the irreducible S -module W , it follows that $\operatorname{res}_S^T \operatorname{ind}_S^T W$ is a completely reducible S -module. Since $[T : S]$ is finite and $\operatorname{char}(F) \nmid [T : S]$, a sharpened version of Maschke's theorem (cf. [CR, §10.20, Problem 8]) ensures that $\operatorname{ind}_S^T W$ is a completely reducible T -module. □

Definition 6.4. Let T be a group. We say that a subgroup S of T is *strongly ascendant* (relative to $\operatorname{char}(F)$) if there is well-ordered set (X, \leq) (not to be confused with the order used in §2), with first element x_0 and last element x_1 , as well as subgroups S_x , $x \in X$, of T such that: $S_{x_0} = S$; $S_{x_1} = T$; S_x is normal of finite index in $S_{x'}$, with $\operatorname{char}(F) \nmid [S_{x'} : S_x]$, for every $x \in X$, $x \neq x_1$, with successor $x' \in X$; if $x \in X$, $x \neq x_0$, and x has no immediate predecessor in X , then $S_x = \bigcup_{y < x} S_y$.

Theorem 6.5. *Keep all of the hypotheses of Theorem 6.1 and assume, in addition, that $[I : H]$ is finite and not divisible by $\text{char}(F)$. Then*

(a) $\text{ind}_H^I W$ is a completely reducible I -module of finite length $\leq [I : H]$ and

$$\text{ind}_H^I W = m_1 V_1 \oplus \cdots \oplus m_t V_t,$$

where $\{V_1, \dots, V_t\}$ is a full set of representatives for the isomorphism classes of irreducible I -modules lying over W . Moreover, each m_i satisfies

$$d(i) = (W, \text{res}_H^I V_i)_H = m_i (V_i, V_i)_I,$$

where $d(i)$ is the (finite) length of the homogeneous H -module $\text{res}_H^I V_i$, that is,

$$\text{res}_H^I V_i \cong W \oplus \cdots \oplus W \quad (d(i) \text{ summands}).$$

(b) We have

$$\text{ind}_H^G W \cong m_1 \text{ind}_I^G V_1 \oplus \cdots \oplus m_t \text{ind}_I^G V_t,$$

where

$$(\text{ind}_I^G V_i, \text{ind}_I^G V_i)_G = (V_i, V_i)_I, \quad 1 \leq i \leq t,$$

$$(\text{ind}_I^G V_i, \text{ind}_I^G V_j)_G = 0, \quad 1 \leq i \neq j \leq t,$$

$$(\text{ind}_I^G V_i, \text{ind}_H^G W) = m_i (V_i, V_i)_I = (W, \text{res}_H^I V_i)_H = d(i), \quad 1 \leq i \leq t.$$

(c) Suppose that at least one of the following conditions holds:

(C1) There is a normal subgroup N of G contained in H and an irreducible N -module W_0 such that W lies over W_0 and $I = I_G(W_0)$.

(C2) G is finite, $\text{char}(F) \nmid |G|$ and $(V_i, V_i)_I = 1$ for all $1 \leq i \leq t$.

(C3) I is a strongly ascendant subgroup of G and $(V_i, V_i)_I = 1$ for all $1 \leq i \leq t$.

Then $\text{ind}_I^G V_1, \dots, \text{ind}_I^G V_t$ are irreducible G -modules. In particular, $\text{ind}_H^G W$ is a completely reducible G -module.

Proof. We know from Proposition 6.3 that $\text{ind}_H^I W$ is a completely reducible I -module. Since $\text{res}_H^I \text{ind}_H^I W$ is the direct sum of $[I : H]$ conjugates of the irreducible H -module W , we see that $\text{res}_H^I \text{ind}_H^I W$ has finite length $[I : H]$, whence $\text{ind}_H^I W$ has finite length $\leq [I : H]$. Thus

$$(6.1) \quad \text{ind}_H^I W = m_1 V_1 \oplus \cdots \oplus m_t V_t,$$

where V_1, \dots, V_t are non-isomorphic irreducible I -modules and each $m_i \geq 1$. Projecting $\text{ind}_H^I W$ onto each component, we see that

$$(\text{ind}_H^I W, V_i)_I \neq 0, \quad 1 \leq i \leq t,$$

so by Frobenius reciprocity

$$(W, \text{res}_H^I V_i)_H \neq 0, \quad 1 \leq i \leq t.$$

Therefore each V_i lies over W . Conversely, if V is an irreducible I -module lying over W then

$$(W, \text{res}_H^I V)_H \neq 0,$$

so

$$(\text{ind}_H^I W, V)_I \neq 0$$

by Frobenius reciprocity. Hence (6.1) implies the existence of one (and only one) i such that $1 \leq i \leq t$ and

$$(V_i, V)_I \neq 0.$$

Since V_i and V are irreducible I -modules, Schur's Lemma yields $V_i \cong V$. Thus $\{V_1, \dots, V_t\}$ is a full set of representatives for the isomorphism classes of irreducible I -modules lying over W . Frobenius reciprocity and (6.1) now give

$$(6.2) \quad (W, \text{res}_H^I V_i)_H = (\text{ind}_H^I W, V_i)_I = (m_i V_i, V_i)_I = m_i (V_i, V_i)_I, \quad 1 \leq i \leq t.$$

Since $H \trianglelefteq I$, with $[I : H]$ finite, and W is an irreducible submodule of $\text{res}_H^I V_i$, Clifford's theory ensures that $\text{res}_H^I V_i$ is the direct sum $d(i)$ conjugates of W , where $d(i)$ is the (finite) length of $\text{res}_H^I V_i$ as an H -module. But W is I -invariant, so $\text{res}_H^I V_i$ is the direct sum of $d(i)$ copies of W . Thus

$$(6.3) \quad (W, \text{res}_H^I V_i)_H = d(i), \quad 1 \leq i \leq t.$$

Let S be any subgroup of G containing I . From (6.1) we obtain

$$(6.4) \quad \begin{aligned} \text{ind}_H^S W &\cong \text{ind}_I^S \text{ind}_H^I W \\ &\cong \text{ind}_I^S (m_1 V_1 \oplus \dots \oplus m_t V_t) \\ &\cong m_1 \text{ind}_I^S V_1 \oplus \dots \oplus m_t \text{ind}_I^S V_t. \end{aligned}$$

From Theorem 6.1, (6.1) and Schur's Lemma we get

$$(6.5) \quad [I : H] = m_1^2 (V_1, V_1)_I + \dots + m_t^2 (V_t, V_t)_I.$$

Setting $V'_i = \text{ind}_I^S V_i$ for $1 \leq i \leq t$, Theorem 6.1 and (6.4) yield

$$(6.6) \quad [I : H] = m_1^2 (V'_1, V'_1)_S + \dots + m_t^2 (V'_t, V'_t)_S + \sum_{i \neq j} m_i m_j (V'_i, V'_j)_S.$$

Now, by Frobenius reciprocity

$$(V'_i, V'_i)_S = (V_i, \text{res}_I^S V'_i)_I, \quad 1 \leq i \leq t.$$

Since V_i is an I -submodule of $\text{res}_I^S V'_i$, it follows that

$$(6.7) \quad (V'_i, V'_i)_S \geq (V_i, V_i)_I, \quad 1 \leq i \leq t.$$

Combining (6.5), (6.6) and (6.7) we infer

$$(6.8) \quad (V'_i, V'_i)_S = (V_i, V_i)_I \text{ and } (V'_i, V'_j)_S = 0, \quad 1 \leq i \neq j \leq t.$$

Making use of (6.4) and (6.8) yields

$$(6.9) \quad (V'_i, \text{ind}_H^S W)_S = (V'_i, m_i V'_i)_S = m_i (V'_i, V'_i)_S = m_i (V_i, V_i)_I, \quad 1 \leq i \leq t.$$

Thus (6.2), (6.3) and (6.9) give

$$(\text{ind}_I^S V_i, \text{ind}_H^S W)_S = (W, \text{res}_H^I V_i)_H = d(i), \quad 1 \leq i \leq t.$$

Suppose (C1) holds. Since each V_i is an irreducible I -module lying over the irreducible N -module W_0 and $I = I_G(W_0)$, it follows from Clifford's Correspondence (cf. Theorem 3.1) that each $\text{ind}_I^G V_i$ is an irreducible G -module.

Suppose (C2) holds. Then, by Maschke's theorem, each $\text{ind}_I^G V_i$ is a completely reducible G -module. Moreover, by (6.8), we have $(\text{ind}_I^G V_i, \text{ind}_I^G V_i)_G = 1$, so each $\text{ind}_I^G V_i$ is irreducible.

Suppose (C3) holds. Let (X, \leq) and $(S_x)_{x \in X}$ be as in Definition 6.4 and set $V = V_i$, where $1 \leq i \leq t$. Then $\text{ind}_I^{S_{x_0}} V = V$ is irreducible. Let $x_0 < x$ and suppose $\text{ind}_I^{S_y} V$ is irreducible for every $y < x$.

Case 1. x has an immediate predecessor y . Since

$$\text{ind}_I^{S_x} V \cong \text{ind}_{S_y}^{S_x} \text{ind}_I^{S_y} V, \quad 1 \leq i \leq t,$$

it follows from Proposition 6.3 that $\text{ind}_I^{S_x} V$ is a completely reducible S_x -module.

But, by above, $(\text{ind}_I^{S_x} V, \text{ind}_I^{S_x} V)_{S_x} = 1$, so $\text{ind}_I^{S_x} V$ is irreducible.

Case 2. x has no immediate predecessor. We readily see that there is a system of representatives Z for the left cosets of I in S_x such that $Z \cap S_y$ is a system of representatives for the left cosets of I in S_y for each $y \leq x$. We have

$$\text{ind}_I^{S_x} V = \bigoplus_{z \in Z} zV.$$

Let u be an arbitrary non-zero element of $\text{ind}_I^{S_x} V$ and let $v \in \text{ind}_I^{S_x} V$. Then there is a finite subset \hat{Z} of Z such that

$$u, v \in \bigoplus_{z \in \hat{Z}} zV.$$

Since $S_x = \bigcup_{y < x} S_y$, there is $y < x$ such that $\hat{Z} \subseteq S_y$. Then

$$u, v \in \bigoplus_{z \in S_y \cap Z} zV \cong \text{ind}_I^{S_y} V.$$

Since $\text{ind}_I^{S_y} V$ is irreducible, $v \in FS_y \cdot u$, whence $v \in FS_x \cdot u$. This proves that $\text{ind}_I^{S_x} V$ is irreducible.

By transfinite induction, $\text{ind}_I^{S_x} V$ is irreducible for every $x \in X$. Since $G = S_{x_1}$, the proof is complete. \square

Note 6.6. It is false, in general, that if S is a strongly ascendant subgroup of T and W is an irreducible S -module then $\text{ind}_S^T W$ is completely reducible. Indeed, let T be the direct product of countably many copies of any finite non-trivial group P and let S be the trivial subgroup of T with trivial S -module W . Suppose $\text{char}(F) \nmid |P|$. Then $\text{ind}_{S_i}^{S_i} W$ is completely reducible for every i (where S_i is the direct product of i copies of P), but $V = \text{ind}_S^T W$ is the regular module of the infinite group T and hence is not completely reducible (the epimorphism $V \rightarrow F$ shows that a supposed complement to the augmentation ideal must be trivial, but V has no trivial submodule).

As a consequence of Theorem 6.5 we obtain the following extension of a well-known irreducibility criterion due to Mackey [Ma], originally proved in the context of finite groups and finite dimensional modules over an algebraically closed field.

Theorem 6.7. *Let $H \leq G$ be groups and let W be an irreducible H -module satisfying*

$$(W, W)_H = 1$$

and

$$(W, {}^x W)_{H \cap x H x^{-1}} = 0, \quad x \in G \setminus H.$$

Suppose at least one of the following conditions hold:

(D1) *There is a normal subgroup N of G contained in H and an irreducible N -module W_0 such that W lies over W_0 and $H = I_G(W_0)$ (this is automatic if $H \trianglelefteq G$).*

(D2) *G is finite and $\text{char}(F) \nmid |G|$.*

(D3) H is a strongly ascendant subgroup of G .

Then $\text{ind}_H^G W$ is an irreducible G -module and $(\text{ind}_H^G W, \text{ind}_H^G W)_G = 1$.

7. PREPARATION FOR MACKEY THEORY

Notation 7.1. Let

$$\Omega = \{(\alpha, \gamma) \in \Phi \mid \exists (\alpha, \beta), (\gamma, \delta) \in D \text{ such that } \alpha < \gamma < \beta < \delta\},$$

$$\Gamma_1 = \Gamma \cup \Omega \text{ (disjoint union).}$$

Note 7.2. Since D is always finite, so is the subset Ω of Φ .

Lemma 7.3. Γ_1 is a closed subset of Φ .

Proof. This is essentially proven in [A3, Proposition 2]. \square

Notation 7.4. Let

$$I = M(\Gamma_1).$$

Theorem 7.5. (a) H is normal in I .

(b) $[I : H] = |R|^{| \Omega |}$. Thus, I/H is a finite group provided R is a finite ring.

(c) I stabilizes λ .

(d) Given any $g \in M \setminus I$ there is $h \in H \cap gHg^{-1}$ such that $\lambda(h) \neq \lambda(g^{-1}hg)$.

Proof. (a) This is included in the proof of [A3, Proposition 2].

(b) This follows from Proposition 2.5.

(c) By Proposition 2.3 it suffices to show that, given $(\alpha, \gamma) \in \Omega$ and $r \in R$, the conjugate character of λ by $1 + re_{\alpha\gamma}$ equals λ . By Proposition 2.3 this verification can be restricted to

$$(7.1) \quad \lambda((1 + re_{\alpha\gamma})(1 + se_{\pi\rho})(1 - re_{\alpha\gamma})) = \lambda(1 + se_{\pi\rho})$$

for all $(\pi, \rho) \in \Gamma$ and $s \in R$. Since $(\alpha, \gamma) \in \Omega$ there exist $(\alpha, \beta), (\gamma, \delta) \in D$ such that

$$\alpha < \gamma < \beta < \delta.$$

Two cases arise:

CASE 1. $\pi = \gamma$. In this case

$$(1 + re_{\alpha\gamma})(1 + se_{\gamma\rho})(1 - re_{\alpha\gamma}) = (1 + se_{\gamma\rho})(1 + rse_{\alpha\rho}),$$

so we need to check that

$$\lambda(1 + rse_{\alpha\rho}) = 1.$$

This is automatically true of $(\alpha, \rho) \notin D$. But (α, ρ) cannot be in D , for in that case $\rho = \beta$, whence $(\gamma, \beta) \notin \Gamma$.

CASE 2. $\rho = \alpha$. In this case

$$(1 + re_{\alpha\gamma})(1 + se_{\pi\alpha})(1 - re_{\alpha\gamma}) = (1 + se_{\pi\alpha})(1 - sre_{\pi\gamma}),$$

so we need to check that

$$\lambda(1 - sre_{\pi\gamma}) = 1.$$

This is automatically true of $(\pi, \gamma) \notin D$. But (π, γ) cannot be in D , for in that case $(\pi, \alpha) \notin \Gamma$.

(d) This is essentially contained in the proof of [A3, Proposition 1]. However, due to its critical role and technical nature, we reproduce André's argument, suitably modified to our purposes.

Let $g \in M \setminus I$. Then $g = 1 + x$, where

$$x = \sum_{(\alpha, \beta) \in \Phi} x_{\alpha\beta} e_{\alpha\beta},$$

where all $x_{\alpha\beta} \in R$ and almost all of them are equal to 0. Since $g \notin I$, the set

$$A = \{(\alpha, \beta) \in \Phi \setminus \Gamma_1 \mid x_{\alpha\beta} \neq 0\}$$

is non-empty. To any $(\alpha, \beta) \in A$ there corresponds a unique $(\alpha, \gamma) \in D$. Thus,

$$D_1 = \{(\alpha, \gamma) \in D \mid \exists (\alpha, \beta) \in \Phi \setminus \Gamma_1 \text{ such that } x_{\alpha\beta} \neq 0\}$$

is non-empty. Choose $(\alpha, \gamma) \in D_1$ with γ as large as possible. This can be done, because D_1 is a finite, totally ordered, non-empty set.

It follows that if $(\rho, \sigma) \in D$, $\gamma < \sigma$, and $x_{\rho\tau} \neq 0$ for some $\rho < \tau < \sigma$ (so that $(\rho, \tau) \notin \Gamma$), then necessarily $(\rho, \tau) \in \Omega$ (for otherwise $(\rho, \tau) \notin \Gamma_1$, against the choice of (α, γ)) (*).

For this choice of $(\alpha, \gamma) \in D_1$ there exists $(\alpha, \beta) \in \Phi \setminus \Gamma_1$ such that $x_{\alpha\beta} \neq 0$.

We claim that for all $r \in R$, we have

$$g(1 + re_{\beta\gamma})g^{-1} \in H \cap gHg^{-1}.$$

Indeed, since $(\alpha, \gamma) \in D$ and $(\alpha, \beta) \notin \Gamma$, we have $\alpha < \beta < \gamma$. Moreover, since $(\alpha, \beta) \notin \Omega$, for any $\delta \in \Lambda$

$$(7.2) \quad \gamma < \delta \Rightarrow (\beta, \delta) \notin D.$$

If $(\beta, \gamma) \notin \Gamma$ there would be $\delta \in \Gamma$ such that $\beta < \gamma < \delta$ and $(\beta, \delta) \in D$, against (7.2). This shows that $(\beta, \gamma) \in \Gamma$, so $1 + re_{\beta\gamma} \in H$, whence $g(1 + re_{\beta\gamma})g^{-1} \in gHg^{-1}$.

Next we show that

$$g(1 + re_{\beta\gamma})g^{-1} = (1 + x)(1 + re_{\beta\gamma})(1 + x)^{-1} = (1 + x)(1 + re_{\beta\gamma})(1 + y) \in H,$$

where $(1 + x)^{-1} = 1 + y$ for a unique $y \in J$ (the nil ring J was defined in §2). Now

$$(7.3) \quad (1 + re_{\beta\gamma})(1 + y) = 1 + y + re_{\beta\gamma} + re_{\beta\gamma}y.$$

Since $(\beta, \gamma) \in \Gamma$, it follows that $(\beta, \delta) \in \Gamma$ for every $\delta \in \Lambda$ such that $\gamma \leq \delta$. Thus, the multiplication in J implies

$$(7.4) \quad 1 + re_{\beta\gamma} + re_{\beta\gamma}y \in H.$$

Multiplying (7.3) on the left by $(1 + x)$ we find

$$(7.5) \quad g(1 + re_{\beta\gamma})g^{-1} = 1 + re_{\beta\gamma} + re_{\beta\gamma}y + xre_{\beta\gamma} + xre_{\beta\gamma}y.$$

Suppose, if possible, that for some $(\rho, \pi) \notin \Gamma$, the (ρ, π) coefficient of $xre_{\beta\gamma}$ is not 0. Then $\pi = \gamma$ and $\rho < \beta$, in which case the coefficient is $x_{\rho\beta}r \neq 0$. Since $(\rho, \gamma) = (\rho, \pi) \notin \Gamma$, there exists $(\rho, \sigma) \in D$ with $\rho < \gamma < \sigma$. As $x_{\rho\beta} \neq 0$ and $\rho < \beta < \gamma < \sigma$, (*) implies $(\rho, \beta) \in \Omega$. Then there exists $(\beta, \delta) \in D$ such that $\rho < \beta < \sigma < \delta$. Since $\gamma < \sigma$, we infer $\gamma < \delta$, against (7.2). This proves that $xre_{\beta\gamma}$ is in the R -span of Γ . As above, this implies that $xre_{\beta\gamma}y$ is in the R -span of Γ . Combining this with (7.4) proves the claim.

We next claim that there is $r \in R$ such that $h_r = g(1 + re_{\beta\gamma})g^{-1} \in H \cap gHg^{-1}$ satisfies $\lambda(h_r) \neq \lambda(g^{-1}h_rg)$. Since $\alpha \neq \beta$ and $(\alpha, \gamma) \in D$, it follows that $(\beta, \gamma) \notin D$, whence

$$\lambda(g^{-1}h_rg) = \lambda(1 + re_{\beta\gamma}) = 1, \quad r \in R.$$

Thus, we are reduced to showing the existence of $r \in R$ such that

$$(7.6) \quad \lambda(h_r) \neq 1.$$

Now $(\alpha, \gamma) \in D$ means $(\alpha, \gamma) = (\alpha_i, \beta_i)$ for a unique $1 \leq i \leq m$. As λ_i is right primitive and $x_{\alpha\beta} \neq 0$, there is $r \in R$ such that $\lambda_i(x_{\alpha\beta}r) \neq 1$, that is,

$$(7.7) \quad \lambda(1 + x_{\alpha\beta}re_{\alpha\gamma}) \neq 1.$$

We claim that (7.6) holds for this choice of r . Indeed, when we express h_r , namely (7.5), in the canonical form (2.4), the non-trivial factors must be of one of the following forms:

- (i) $1 + re_{\beta\gamma}$,
- (ii) $1 + ry_{\gamma\delta}e_{\beta\delta}$, with $\gamma < \delta$,
- (iii) $1 + x_{\alpha\beta}re_{\alpha\gamma}$,
- (iv) $1 + x_{\rho\beta}re_{\rho\gamma}$, with $\alpha \neq \rho < \beta$, or
- (v) $1 + x_{\rho\beta}ry_{\gamma\delta}e_{\rho\delta}$, with $\rho < \beta < \gamma < \delta$.

Since λ is a linear character of H , the value of λ on h_r is equal to the product of its values on the non-trivial factors above. In view of (7.7), it remains to show that λ has value 1 on all non-trivial factors of type different from (iii). This is clear for types (i), (ii) and (iv) since $(\rho, \gamma), (\beta, \gamma) \notin D$ (as $\beta, \rho \neq \alpha$) and $(\beta, \delta) \notin D$ (by (7.2)). Suppose, if possible, that a factor of type (v) is not trivial (so, in particular, $x_{\rho\beta} \neq 0$) with $(\rho, \delta) \in D$. Then, by (*), $(\rho, \beta) \in \Omega$. Thus there is $\epsilon \in \Lambda$ such $(\beta, \epsilon) \in D$ and $\rho < \beta < \delta < \epsilon$. But $\gamma < \epsilon$, so (7.2) is contradicted. \square

8. DECOMPOSITION OF A BASIC MODULE

Theorem 8.1. *We have*

$$(V(D, f), V(D, f))_M = [I : H] = |R|^{|\Omega|}.$$

Proof. Immediate consequence of Theorems 5.4, 6.1 and 7.5. \square

Note 8.2. Theorem 8.1 is valid for arbitrary (Λ, \leq) and R , not necessarily finite.

Theorem 8.3. *The following conditions are equivalent:*

- (a) $V(D, f)$ is irreducible.
- (b) D is non-overlapping.
- (c) $\Omega = \emptyset$.
- (d) $I = H$.
- (e) $(V(D, f), V(D, f))_M = 1$.

Proof. This follows from Theorems 5.13 and 8.1. \square

Definition 8.4. We will say that F is a *splitting field for I over λ* if $(V, V)_I = 1$ for every irreducible I -module lying over λ .

Note 8.5. We know from Theorem 6.5 that an irreducible I -module V lies over λ if and only if V is a constituent of the finite dimensional, completely reducible module $\text{ind}_H^I W$. Thus, there are finitely many irreducible I -modules lying over λ . Since $[I : H]$ and $\dim(W)$ are finite, every irreducible I -module lying over λ is finite dimensional. Thus there is a finite extension K of F that is a splitting field for I over λ (let K be the subfield of \overline{F} generated by the entries of the matrix representations associated to each isomorphism type of irreducible I -module lying over λ). Thus, there is not much loss of generality in assuming that F itself is a splitting field for I over λ .

Note 8.6. Suppose that λ is extendible to I (see Theorem 8.14 for the exact conditions when this happens). Since $(W, W)_H = 1$, it follows from Lemma 3.2 and Theorem 3.4 that F is a splitting field for I over λ provided F is a splitting field for I/H in the usual sense. In Example 8.19, λ is not extendible to I but, nevertheless, a splitting field for I/H is a splitting field for I over λ . This implication may be true in general, a matter that will not be discussed further.

Theorem 8.7. *Let $\alpha = \min\{\alpha_i \mid 1 \leq i \leq m\}$ and $\beta = \max\{\beta_i \mid 1 \leq i \leq m\}$. Suppose the closed interval $[\alpha, \beta]$ of Λ is well-ordered under \geq , the inverse order of \leq . Assume, in addition, that R is finite. Then I is a strongly ascendant subgroup of M .*

Proof. Let Ψ be a closed subset of Φ satisfying the following conditions:

- (A1) There is a first $\delta \in (\alpha, \beta]$ under \geq such that $M((\rightarrow, \delta]) \subseteq M(\Psi)$.
- (A2) If $\gamma = \delta'$ is the immediate successor of δ in $[\alpha, \beta]$ under \geq , then

$$\Psi^\# = \{(\sigma, \gamma) \in \Phi \mid (\sigma, \gamma) \notin \Psi\}$$

is a finite subset of Φ .

An example is $\Psi = \Gamma_1$, in which case (A1) holds with $\delta = \beta$ and $|\Psi^\#| \leq 1$.

In any case, either $\Psi^\# = \emptyset$, and we set $\Psi' = \Psi$, or else

$$(8.1) \quad \Psi^\# = \{(\sigma_1, \gamma), \dots, (\sigma_n, \gamma)\},$$

$$\sigma_1 < \dots < \sigma_n,$$

in which case the following are all closed subsets of Φ , each normal in the next:

$$\Psi(0) = \Psi \subset \Psi(1) = \Psi \cup \{(\sigma_1, \gamma)\} \subset \dots \subset \Psi(n) = \Psi \cup \{(\sigma_1, \gamma), \dots, (\sigma_n, \gamma)\}.$$

Moreover, by Proposition 2.5,

$$[M(\Psi(i+1)) : M(\Psi(i))] = |R|, \quad 0 \leq i < n,$$

where $\text{char}(F) \nmid |R|$ by Lemma 4.2, and $\Psi' = \Psi(n)$ satisfies (A1), (A2) as well as

$$(8.2) \quad M((\rightarrow, \gamma]) \subseteq M(\Psi').$$

Define the family $(\Psi_\gamma)_{\gamma \in [\alpha, \beta]}$ of closed subsets of Ψ as follows:

$$\Psi_\beta = \Gamma_1, \quad \Psi_\gamma = (\Psi_\delta)' \text{ if } \gamma = \delta', \quad \Psi_\gamma = \bigcup_{\delta > \gamma} \Psi_\delta \text{ if } \gamma \text{ has no immediate } \geq\text{-predecessor.}$$

This is possible because, as D is finite, so is (8.1) when $\Psi = \Psi_\delta$ for any $\delta \in (\alpha, \beta]$. By construction, Ψ_δ is subnormal in $\Psi_{\delta'}$, with $[M(\Psi_{\delta'}) : M(\Psi_\delta)]$ finite and not divisible by $\text{char}(F)$ for every $\delta \in (\alpha, \beta]$. Moreover, (8.2) implies $M = M(\Psi_\alpha)$. \square

Theorem 8.8. *Suppose R is finite and F is a splitting field for I over λ . Let*

$$\text{ind}_H^I W = m_1 V_1 \oplus \dots \oplus m_t V_t,$$

be the decomposition of $\text{ind}_H^I W$ ensured by part (a) of Theorem 6.5 (which applies, due to Lemma 4.2 and parts (a), (b) and (c) of Theorem 7.5), where $\{V_1, \dots, V_t\}$ is a full set of representatives for the isomorphism classes of irreducible I -modules lying over W , and each m_i satisfies

$$(W, \text{res}_H^I V_i)_H = \dim(V_i) = m_i.$$

Assume, in addition, that I is a strongly ascendant subgroup of M . Then

$$V(D, f) \cong m_1 \text{ind}_I^M V_1 \oplus \dots \oplus m_t \text{ind}_I^M V_t,$$

where $\text{ind}_I^M V_1, \dots, \text{ind}_I^M V_t$ are non-isomorphic irreducible M -modules, and

$$(\text{ind}_I^M V_i, V(D, f))_M = \dim(V_i) = m_i, \quad 1 \leq i \leq t.$$

Proof. Use Theorems 5.4, 6.5 and 7.5(d). \square

Note 8.9. Suppose R is finite. In view of Theorem 8.7, all we have to do to ensure that every I is a strongly ascendant subgroup of M is to start with any well-order and impose its inverse on Λ . This is automatic when Λ is finite, which is just a very special, albeit important, case.

Observe that I need not be a strongly ascendant subgroup of M in general. Suppose, for instance, that Λ is a closed interval $[\alpha, \beta]$ and β is an accumulation point when $[\alpha, \beta]$ is given the topology of the strict order $<$ associated to \leq (cf. [K, Chapter 1, Problem I]). If $D = \{(\alpha, \beta)\}$, then $I = H$ is self-normalizing.

Note 8.10. Suppose R is finite, $M = U_n(R)$ and F is a splitting field for (the finite group) I . Then condition (C2) of Theorem 6.5 is satisfied, and the conclusion of Theorem 8.8 follow automatically, that is, without resorting to strongly ascendant subgroups.

Note 8.11. Let

$$\Gamma_0 = \Gamma \setminus \{(\sigma, \tau) \in \Gamma \mid \exists i \text{ such that } 1 \leq i \leq m \text{ and } \alpha_i < \sigma < \tau < \beta_i\}.$$

It is easy to see that $N = M(\Gamma_0)$ satisfies

$$N = \text{core}_M(H).$$

By Theorem 7.5(c), we have $I \subseteq I_M(\lambda|_N)$, although equality is not necessarily true in general. However, it does occur, occasionally. Suppose, for instance, that, if $1 \leq i < m$, then α_{i+1} is the only element of Λ satisfying $\alpha_i < \alpha_{i+1} < \beta_i$. Then $I = I_M(\lambda|_N)$ (the case $m = 1$ is treated in §4). Whatever the example, suppose that $I = I_M(\lambda|_N)$. Then condition (C1) of Theorem 6.5 is satisfied. Thus, if R is finite and F is a splitting field for I over λ , then the conclusion of Theorem 8.8 is achieved without involving strongly ascendant subgroups.

Definition 8.12. We will say that D has a special triple if it contains a subset $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$ satisfying:

$$\alpha_1 < \alpha_2 < \alpha_3 = \beta_1 < \beta_2 < \beta_3.$$

Proposition 8.13. $[\Gamma_1, \Gamma_1] \cap D = [\Omega, \Omega] \cap D$. Moreover, these are non-empty if and only if D contains a special triple.

Proof. Suppose $(\alpha, \beta) \in [\Gamma_1, \Gamma_1] \cap D$. Then, by definition, there is a chain

$$(\gamma_1, \gamma_2), \dots, (\gamma_{n-1}, \gamma_n) \in \Gamma_1, \quad n \geq 3,$$

such that $\gamma_1 = \alpha$ and $\gamma_n = \beta$. Since $\Gamma_1 \subseteq \Phi$, we have

$$\alpha = \gamma_1 < \gamma_2 < \dots < \gamma_{n-1} < \gamma_n = \beta.$$

In particular, $\alpha < \gamma_2 < \beta$, so $(\alpha, \gamma_2) \notin \Gamma$ and therefore $(\alpha, \gamma_2) \in \Omega$. By definition, there exists $(\alpha_2, \beta_2) \in D$ such that

$$\alpha < \alpha_2 < \beta < \beta_2, \quad \gamma_2 = \alpha_2.$$

From

$$\alpha_2 < \gamma_3 \leq \beta < \beta_2,$$

we infer $(\alpha_2, \gamma_3) \notin \Gamma$, hence $(\alpha_2, \gamma_3) \in \Omega$. By definition, there exists $(\alpha_3, \beta_3) \in D$ such that

$$\alpha_2 < \alpha_3 < \beta_2 < \beta_3, \quad \gamma_3 = \alpha_3.$$

If $\alpha_3 = \beta$ then

$$\alpha < \alpha_2 < \alpha_3 = \beta < \beta_2 < \beta_3$$

and we are done. Otherwise,

$$\alpha_3 < \gamma_4 \leq \beta < \beta_3,$$

implies, as before, the existence of $(\alpha_4, \beta_4) \in D$ such that

$$\alpha_3 < \alpha_4 < \beta_3 < \beta_4, \quad \gamma_4 = \alpha_4.$$

If $\alpha_4 = \beta$ then

$$\alpha < \alpha_2 < \alpha_3 < \alpha_4 = \beta < \beta_2 < \beta_3 < \beta_4$$

and we are done. Otherwise continue this process to obtain the desired result. \square

Theorem 8.14. *The following conditions are equivalent:*

- (a) λ is extendible to I .
- (b) $[\Gamma_1, \Gamma_1] \cap D = \emptyset$.
- (c) $[\Omega, \Omega] \cap D = \emptyset$.
- (d) D has no special triple.

Proof. Suppose first $(\alpha, \beta) \in [\Gamma_1, \Gamma_1] \cap D$. By Proposition 2.7, $M_{\alpha\beta} \subseteq [I, I]$, so any group homomorphism $I \rightarrow F^*$ is trivial on $M_{\alpha\beta}$, whence λ is not extendible to I .

Suppose next $[\Gamma_1, \Gamma_1] \cap D = \emptyset$. Then $D \subseteq \Gamma_1 \setminus [\Gamma_1, \Gamma_1]$. We certainly have a group homomorphism from the external direct product of all $M_{\alpha\beta}$, $(\alpha, \beta) \in \Gamma_1 \setminus [\Gamma_1, \Gamma_1]$,

$$\mu : \prod_{(\alpha, \beta) \in \Gamma_1 \setminus [\Gamma_1, \Gamma_1]} M_{\alpha\beta} \rightarrow F^*$$

that agrees with λ_i on $M_{\alpha_i\beta_i}$ for every $1 \leq i \leq m$. It follows from Proposition 2.7 that μ gives rise to an extension of λ to I .

This proves the equivalence between (a) and (b). Now apply Proposition 8.13. \square

Theorem 8.15. *Suppose $|R|$ is finite, F is a splitting field for $S = I/H$ and D has no special triple. Assume, in addition, that I is a strongly ascendant subgroup of M . Then $V(D, f)$ has the following decomposition as the direct sum of irreducible non-isomorphic M -modules with indicated multiplicities:*

$$V(D, f) \cong \bigoplus_{\chi \in \text{Irr}(S)} \chi(1) \text{ind}_I^M(U_\chi \otimes W_1),$$

where U_χ is an irreducible S -module -viewed as an I -module- affording χ , and W_1 is the vector space W acted upon via an extension of λ to I .

Proof. Use Note 8.6 as well as Theorems 3.4, 8.8 and 8.14. \square

Corollary 8.16. *Suppose that R is finite, D has no special triple, and $S = I/H$ is abelian (this is equivalent to $[\Omega, \Omega] \subseteq \Gamma$). Assume, in addition, that I is a strongly ascendant subgroup of M . Then $V(D, f)$ has the following decomposition into non-isomorphic M -modules:*

$$V(D, f) \cong \bigoplus_{\chi \in \text{Irr}(S)} \text{ind}_I^M W_1(\chi),$$

where $W_1(\chi)$ is the vector space W acted upon I via the only extension of λ to I that satisfies $1 + re_{\sigma\tau} \mapsto \chi(r)$ for all $(\sigma, \tau) \in \Omega$.

Proof. Use Lemma 4.3 and Theorem 8.15. \square

Definition 8.17. We will say that D is overlapping of type 1 provided the following conditions are satisfied:

- $\alpha_1 < \dots < \alpha_m$ (this can always be arranged), where $m \geq 2$.
- The intervals $(\alpha_i, \beta_i), (\alpha_j, \beta_j)$ overlap if and only if $|i - j| = 1$;
- No β_i equals an α_j (this is automatic if $|D| = 2$).

Let $\Lambda = \mathbb{N}$ under its usual order. Then $\{(1, 3), (2, 5), (4, 7), (6, 8)\}$ is an overlapping subset of Φ of type 1, whereas $\{(1, 3), (2, 4), (3, 5)\}$ and $\{(1, 4), (2, 5), (3, 6)\}$ are not.

As the following result indicates, a family of examples to which Corollary 8.16 applies is given by the overlapping subsets of Φ of type 1.

Lemma 8.18. *Suppose D is an overlapping subset of Φ of type 1. Then D has no special triples and I/H is abelian.*

Proof. By Definition 8.17, we have $\Omega = \{(\alpha_1, \alpha_2), \dots, (\alpha_{m-1}, \alpha_m)\}$, $[\Omega, \Omega] \subseteq \Gamma$ and D has no special triples. \square

Example 8.19. *Suppose R is finite, D satisfies*

$$\alpha_1 < \alpha_2 < \dots < \alpha_m = \beta_1 < \beta_2 < \dots < \beta_m, \quad m \geq 3,$$

and F is a splitting field for $U_{m-2}(R)$. Assume, in addition, that $[\alpha_1, \beta_m]$ is well-ordered by \geq . Then $V(D, f)$ has the following decomposition as the direct sum of irreducible non-isomorphic M -modules with indicated multiplicities:

$$V(D, f) \cong \bigoplus_{\chi \in \text{Irr}(B)} \dim(U_\chi) \dim(V) \text{ind}_I^M(U_\chi \otimes V).$$

Here $I = T \rtimes B$ for suitable subgroups $B \cong U_{m-2}(R)$ and T described below; each U_χ is an irreducible B -module -viewed as an I -module- affording χ ; V is an irreducible I -module of dimension $|R|^{m-2}$, as described below; each $U_\chi \otimes V$ is an irreducible I -module. Moreover, when $m = 3$ we have $V(D, f) \cong |R| \text{ind}_I^M V$.

Proof. By hypothesis

$$\Omega = \{(\alpha_1, \alpha_2), \dots, (\alpha_1, \alpha_{m-1})\} \cup \{(\alpha_2, \alpha_m), \dots, (\alpha_{m-1}, \alpha_m)\} \cup \Omega'$$

where

$$\Omega' = \{(\alpha_i, \alpha_j) \mid 1 < i < j < m\}.$$

Let

$$K = I \cap (M((\rightarrow, \beta_1))M((\alpha_1, \leftarrow))),$$

$$A = I \cap M([\alpha_1, \beta_1]),$$

$$B = I \cap M((\alpha_1, \beta_1)),$$

$$L = M_{\alpha_1 \alpha_2} \cdots M_{\alpha_1 \alpha_{m-1}} M_{\alpha_1 \beta_1} M_{\alpha_2 \alpha_m} \cdots M_{\alpha_{m-1} \alpha_m}.$$

Note that A (resp. B) is the McLain group associated to $\{\alpha_1, \dots, \alpha_m\}$ (resp. $\{\alpha_2, \dots, \alpha_{m-1}\}$). In particular, $B \cong U_{m-2}(R)$. Moreover, B and L are respectively $M((\alpha_1, \beta_1))$ and $M^{\alpha_1 \beta_1}$ for the McLain group associated to $\{\alpha_1, \dots, \alpha_m\}$. Furthermore, we have

$$A = L \rtimes B \text{ and } I = K \rtimes A,$$

so that for

$$T = K \rtimes L,$$

we have

$$I = T \rtimes B.$$

Our description of L and the results of §4 ensure that there is one and only one irreducible L -module X , up to isomorphism, lying over λ_1 (viewed as a linear character of M_{α_1, β_1} via (2.2)). Here $\dim(X) = |R|^{m-2}$ and $(X, X)_L = 1$.

We deduce from §4 and the above interpretation of A , B and L that the action of L on X can be extended to A . Let V be the vector space X viewed as an A -module under this action and let $S : A \rightarrow \mathrm{GL}(V)$ be the associated representation.

On the other hand, since $\Omega \cap (\rightarrow, \beta_1) = \emptyset$, we see that K is a subgroup of H . In particular, λ is defined on K (a critical and subtle point). As $I = K \rtimes A$, we may define $P : I \rightarrow \mathrm{GL}(V)$ by

$$P(ak) = S(a)\lambda(k).$$

By Theorem 7.5, I stabilizes λ , so P is a group homomorphism and V is an irreducible I -module. Since $(X, X)_L = 1$, we have $(V, V)_I = 1$. Moreover, by construction, V lies over W and

$$(W, \mathrm{res}_H^I V)_H = |R|^{m-2} = \dim(V).$$

We claim that an irreducible I -module V' lies over W if and only if $V' \cong U \otimes V$, where U is an irreducible B -module viewed as I -module (and hence acted upon trivially by T).

Indeed, let U be an irreducible B -module viewed as I -module. Here $\mathrm{res}_T^I V$ is irreducible (since so is $\mathrm{res}_L^I X$) and, by definition, U is acted upon trivially by L , so, by Theorem 3.4, $U \otimes V$ is an irreducible I -module. It actually lies over W . In fact, since T acts trivially on U we see that

$$(W, \mathrm{res}_H^I U \otimes V)_H = \dim(U)\dim(V).$$

By Theorem 3.4 the $U \otimes V$ are non-isomorphic and since F is a splitting field for $U_{m-2}(R)$, we have $(U, U)_B = 1$, so by Lemma 3.2

$$(8.3) \quad (U \otimes V, U \otimes V)_I = 1.$$

On the other hand, Theorem 8.1 gives

$$(V(D, f), V(D, f))_M = [I : H].$$

Let

$$Z = \bigoplus_{\chi \in \mathrm{Irr}(B)} \dim(U_\chi) \dim(V) \mathrm{ind}_I^M(U_\chi \otimes V).$$

By Theorem 6.5 and (8.3), for each $\chi \in \mathrm{Irr}(B)$, we have

$$(\mathrm{ind}_I^M(U_\chi \otimes V), \mathrm{ind}_I^M(U_\chi \otimes V))_M = 1,$$

so

$$(Z, Z)_M = \dim(V)^2 \sum_{\chi \in \mathrm{Irr}(B)} \dim(U_\chi)^2 = \dim(V)^2 |B| = |R|^{2(m-2)} |B| = [I : H].$$

But $V(D, f)$ is completely reducible by Theorems 6.5 and 8.7, so $V(D, f) = Z$. \square

Corollary 8.20. *Suppose that R is finite, D has a special triple, and Λ is well-ordered by \geq . Then $V(D, f)$ has a repeated irreducible constituent.*

Proof. Immediate consequence of Example 8.19. \square

Theorem 8.21. *Suppose R is finite and F is a splitting field for I over λ . Assume, in addition, Λ is well-ordered by \geq . Then $V(D, f)$ is multiplicity free if and only if I/H is abelian (this is equivalent to $[\Omega, \Omega] \subseteq \Gamma$) and D has no special triple.*

Proof. If I/H is abelian and D has no special triple then $V(D, f)$ is multiplicity free by Theorems 8.15 and 8.7. If D has a special triple then Corollary 8.20 shows that $V(D, f)$ has multiplicity, while if D has no special triple but I/H is non-abelian then Theorems 8.15 and 8.7 show that $V(D, f)$ has multiplicity. \square

Example 8.22. *Suppose R is finite, Λ is well-ordered by \geq , and D satisfies*

$$\alpha_1 < \alpha_2 < \alpha_3 = \beta_1 < \alpha_4 = \beta_2 < \cdots < \alpha_m = \beta_{m-2} < \beta_{m-1} < \beta_m,$$

where $m \geq 3$. Let n be the largest integer such that $2n \leq m$. Then

(a) If $m = 2n + 1$ there is one and only one irreducible I -module, say V , up to isomorphism, lying over λ . Moreover, $\dim(V) = |R|^n$ with $\text{ind}_I^M V$ is irreducible and

$$V(D, f) \cong |R|^n \text{ind}_I^M V.$$

(b) If $m = 2n$ there are exactly $|R|$ irreducible I -modules, say $V_1, \dots, V_{|R|}$, up to isomorphism, lying over λ . Moreover, each $\dim(V_i) = |R|^{n-1}$, the M -modules $\text{ind}_I^M V_1, \dots, \text{ind}_I^M V_{|R|}$ are irreducible and non-isomorphic, and

$$V(D, f) \cong |R|^{n-1} \text{ind}_I^M V_1 \oplus \cdots \oplus |R|^{n-1} \text{ind}_I^M V_{|R|}.$$

Proof. By hypothesis

$$\Omega = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \dots, (\alpha_{m-1}, \alpha_m)\}.$$

Set

$$\Omega_0 = \{(\alpha_1, \alpha_2), (\alpha_3, \alpha_4), \dots, (\alpha_{2n-1}, \alpha_{2n})\},$$

which is a closed abelian subset of Φ . Let

$$S = \prod_{(\gamma, \delta) \in \Gamma_0} M_{\gamma\delta} \text{ (internal direct product inside } I)$$

and

$$I_0 = M(\Gamma \cup \Omega_0) = H \rtimes S.$$

By Theorem 7.5(c), λ is I -invariant. Thus, given any $\mu \in \text{Hom}(S, F^*)$, the map $\lambda_\mu : M(\Gamma_0) \rightarrow F^*$, given by

$$\lambda_\mu(hs) = \lambda(h)\mu(s), \quad h \in H, s \in S,$$

is a group homomorphism extending λ . We readily verify that $I_0 \trianglelefteq I$ and the inertia group of λ_μ in I is I_0 . Let W_μ be the vector space W acted upon I_0 via λ_μ . Then $V_\mu = \text{ind}_{I_0}^I W_\mu$ is irreducible and $(V_\mu, V_\mu)_I = 1$ by Theorem 3.1. Moreover, it is obvious that V_μ lies over W .

Suppose first $m = 2n + 1$. Then

$$\dim(V_\mu) = [I : I_0] = |R|^n = [I_0 : H].$$

Thus Theorem 6.5(a) implies $\text{ind}_H^I W \cong |R|^n V_\mu$. This proves that the V_μ is independent of μ , up to isomorphism. Now apply Theorems 6.5(b) and 8.7.

Suppose next $m = 2n$. Then

$$\dim(V_\mu) = [I : I_0] = |R|^{n-1} \text{ and } [I_0 : H] = |R|^n.$$

Each λ_μ is conjugate under I to exactly $[I : I_0] = |R|^{n-1}$ extensions of λ , whereas by Lemma 4.3, λ has a total of $|S| = |R|^n$ extensions to I_0 . Thus, these $|R|^n$ extensions fall into exactly $|R|$ classes upon conjugation by I . On the other hand, by Clifford's theory, $V_\mu \cong V_\nu$ if and only if μ and ν are in the same I -conjugacy class. Let $\mu_1, \dots, \mu_{|R|}$ be representatives for the I -conjugacy classes of extensions of λ to I_0 . Since

$$\dim(\text{ind}_H^I W) = [I : H] = |R| \times |R|^{n-1} \times |R|^{n-1},$$

it follows from Theorem 6.5(a) that

$$\text{ind}_H^I W \cong |R|^{n-1} V_{\mu_1} \oplus \dots \oplus |R|^{n-1} V_{\mu_{|R|}}.$$

Now apply Theorems 6.5(b) and 8.7. \square

Example 8.23. Suppose R is finite and $D = D_1 \cup \dots \cup D_\ell$ is a disconnection, where each D_i has size $m_i = 2n_i + 1$ and is as in Example 8.22. Assume, in addition, that Λ is well-ordered by \geq . Let f_i be the restriction of f to D_i , $1 \leq i \leq \ell$. Then

$$V(D, f) \cong |R|^n (Y_1 \otimes \dots \otimes Y_\ell),$$

where $n = n_1 + \dots + n_\ell$, each Y_i is the irreducible M -module from Example 8.22 satisfying $V(D_i, f_i) \cong |R|^{n_i} Y_i$, and $Y_1 \otimes \dots \otimes Y_\ell$ is an irreducible M -module.

Proof. This follows from Theorem 3.6 and Example 8.22. \square

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REFERENCES

- [A] C.A.M. André, *Basic characters of the unitriangular group*, J. Algebra 175 (1995) 287-319.
- [A2] C.A.M. André, *Basic characters of the unitriangular group (for arbitrary primes)*, Proc. Amer. Math. Soc. 130 (2002) 1943-1954.
- [A3] C.A.M. André, *Hecke algebras for the basic characters of the unitriangular group*, Proc. Amer. Math. Soc. 132 (2004) 987-996.
- [A4] C.A.M. André, *Irreducible characters of finite algebra groups*, Matrices and group representations (Coimbra, 1998), 65-80, Textos Mat. Sér. B, 19, Univ. Coimbra, Coimbra, 1999.
- [AN] C.A.M. André and A. Nicolás, *Supercharacters of the adjoint group of a finite radical ring*, J. Group Theory 11 (2008) 709-746.
- [B] M. Boyarchenko, *Representations of unipotent groups over local fields and Gutkin's conjecture*, Math. Res. Lett. 18 (2011) 539-557.
- [CMS] G. Cliff, D. McNeilly and F. Szechtman, *Weil representations of symplectic groups over rings*, J. London Math. Soc. (2) 62 (2000) 423-436.
- [CMS2] G. Cliff, D. McNeilly and F. Szechtman, *Clifford and Mackey theory for Weil representations of symplectic groups*, J. Algebra 262 (2003) 348-379.
- [CR] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.
- [CG] H.L. Claassen and R.W. Goldbach, *A field-like property of finite rings*, Indag. Math. 3 (1992) 11-26.
- [CS] G. Cutolo and H. Smith, *Residually finite subgroups of some countable McLain groups*, Arch. Math. (Basel) 93 (2009) 513-520.
- [DI] D. Diaconis and I.M. Isaacs, *Supercharacters and superclasses for algebra groups*, Trans. Amer. Math. Soc. 360 (2008) 2359-2392.
- [DG] M. Droste and R. Göbel, *McLain groups over arbitrary rings and orderings*, Math. Proc. Cambridge Philos. Soc. 117 (1995) 439-467.
- [Ga] P.X. Gallagher, *Groups characters and normal Hall subgroups*, Nagoya Math. J. 21 (1962) 223-230.
- [G] D. Ginzburg, *On the representations of q unipotent groups over a finite field*, Israel J. Math. 181 (2011) 387-422.

- [GMP] R. Gow, M. Marjoram and A. Previtali, *On the irreducible characters of a Sylow 2-subgroup of the finite symplectic group in characteristic 2*, J. Algebra 241 (2001) 393–409.
- [H] Z. Halasi, *On the characters and commutators of finite algebra groups*, J. Algebra 275 (2004) 481–487.
- [HP] Z. Halasi and P.P. Pálffy, *The number of conjugacy classes in pattern groups is not a polynomial function*, J. Group Theory 14 (2011) 841–854.
- [HH] P. Hall and B. Hartley, *The stability group of a series of subgroups*, Proc. London Math. Soc. (3) 16 (1966) 1–39.
- [Hi] G. Higman, *Enumerating p -groups I. Inequalities*, Proc. London Math. Soc. (3) 10 (1960) 24–30.
- [Ho] T. Honold, *Characterization of finite Frobenius rings*, Arch. Math. (Basel) 76 (2001) 406–415.
- [I] I.M. Isaacs, *Characters of groups associated with finite algebras*, J. Algebra 177 (1995) 708–730.
- [I2] I.M. Isaacs, *Counting characters of upper triangular groups*, J. Algebra 315 (2007) 698–719.
- [K] J.L. Kelley, *General Topology*, Van Nostrand, Toronto, 1955.
- [L] G.I. Lehrer, *Discrete series and the unipotent subgroup*, Compositio Math. 28 (1974) 9–19.
- [La] E. Lamprecht, *Über I -reguläre Ringe, reguläre Ideale und Erklärungsmoduln. I*, Math. Nachr. 10 (1953) 353–382.
- [Le] T. Le, *Counting irreducible representations of large degree of the upper triangular groups*, J. Algebra 324 (2010) 1803–1817.
- [Le2] T. Le, *Supercharacters and pattern subgroups in the upper triangular groups*, Proc. Edinb. Math. Soc. (2) 56 (2013) 177–186.
- [Lo] M. Loukaki, *Counting characters of small degree in upper unitriangular groups*, J. Pure Appl. Algebra 215 (2011) 154–160.
- [Ma] G.W. Mackey, *On induced representations of groups*, Amer. J. Math. 73 (1951) 576–592.
- [Me] E. Marberg, *Combinatorial methods of character enumeration for the unitriangular group*, J. Algebra 345 (2011) 295–323.
- [Me2] E. Marberg, *Heisenberg characters, unitriangular groups, and Fibonacci numbers*, J. Combin. Theory Ser. A 119 (2012) 882–903.
- [Mm] M. Marjoram, *Irreducible characters of Sylow p -subgroups of classical groups*, PhD thesis, National University of Ireland, Dublin, 1997.
- [Mu] U. Meierfrankenfeld, *Ascending subgroups of irreducible finitary linear group*, J. London Math. Soc. (2) 51 (1995) 75–92.
- [M] D.H. McLain, *A characteristically-simple group*, Proc. Cambridge Philos. Soc. 50 (1954) 641–642.
- [M2] D.H. McLain, *Remarks on the upper central series of a group*, Proc. Glasgow Math. Assoc. 3 (1956) 38–44.
- [R] J.E. Roseblade, *The automorphism group of McLain’s characteristically simple group*, Math. Z. 82 1963 267–282.
- [Ro] D.J.S. Robinson, *Finiteness conditions on subnormal and ascendant Abelian subgroups*, J. Algebra 10 (1968) 333–359.
- [Sz] F. Szechtman, *Clifford theory for infinite dimensional modules*, Comm. Algebra, to appear; arXiv:1504.07319.
- [Sz2] F. Szechtman, *Groups having a faithful irreducible representation*, J. Algebra 454 (2016) 292–307.
- [Sz3] F. Szechtman, *Weil representations of symplectic groups over non-principal rings*, Rocky Mt. J. Math. 35 (2005) 309–329.
- [Sze] B. Szegedy, *Characters of the Borel and Sylow subgroups of classical groups*, J. Algebra **267** (2003), 130–136.
- [T] J. Thompson, unpublished, www.math.ufl.edu/fac/thompson.html.
- [VA] A. Vera-López and J.M. Arregi, *Polynomial properties in unitriangular matrices*, J. Algebra 244 (2001) 343–351.
- [VA2] A. Vera-López and J.M. Arregi, *Polynomial properties in unitriangular matrices. II*, J. Group Theory 8 (2005) 701–717.
- [W] J.S. Wilson, *Groups with many characteristically simple subgroups*, Math. Proc. Cambridge Philos. Soc. 86 (1979) 193–197.
- [We] B.A.F. Wehrfritz, *Locally soluble primitive finitary skew linear groups*, Comm. Algebra 23 (1995) 803–817.

- [Wo] J. Wood, *Duality for modules over finite rings and applications to coding theory*, Amer. J. Math. 121 (1999) 555-575.

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